

Affine Jordan cells, logarithmic correlators, and hamiltonian reduction

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Abstract

We study a particular type of logarithmic extension of $SL(2, \mathbb{R})$ Wess-Zumino-Witten models. It is based on the introduction of affine Jordan cells constructed as multiplets of quasi-primary fields organized in indecomposable representations of the Lie algebra $sl(2)$. We solve the simultaneously imposed set of conformal and $SL(2, \mathbb{R})$ Ward identities for two- and three-point chiral blocks. These correlators will in general involve logarithmic terms and may be represented compactly by considering spins with nilpotent parts. The chiral blocks are found to exhibit hierarchical structures revealed by computing derivatives with respect to the spins. We modify the Knizhnik-Zamolodchikov equations to cover affine Jordan cells and show that our chiral blocks satisfy these equations. It is also demonstrated that a simple and well-established prescription for hamiltonian reduction at the level of ordinary correlators extends straightforwardly to the logarithmic correlators as the latter then reduce to the known results for two- and three-point conformal blocks in logarithmic conformal field theory.

Keywords: Logarithmic conformal field theory, Jordan cell, Wess-Zumino-Witten model, Knizhnik-Zamolodchikov equations, hamiltonian reduction.

1 Introduction

In logarithmic conformal field theory (CFT), a primary field may have a so-called logarithmic partner field on which the Virasoro modes do not all act diagonally. If only one logarithmic field is associated to a given primary field, the two fields constitute a so-called conformal Jordan cell of rank two where the rank indicates the number of fields in the cell. We will be concerned with conformal Jordan cells of rank two only. The appearance of such cells is known to lead to logarithmic singularities in the correlators. We refer to [1] for the first systematic study of logarithmic CFT, and to [2, 3, 4] for recent reviews on the subject. An exposition of links to string theory may be found in [5].

The objective of the present work is to introduce and study a particular logarithmic extension of the $SL(2, \mathbb{R})$ Wess-Zumino-Witten (WZW) model. Alternative extensions have appeared in the literature, see [6, 7, 8, 4, 9], for example, but all seem to differ significantly from ours in foundation and approach.

Our construction is based on a generalization of the standard multiplets of Virasoro primary fields organized as spin- j representation. We find that an infinite number of partner fields seem to be required to complete such an indecomposable representation of $sl(2)$, and we refer to these new multiplets as affine Jordan cells.

We consider the case where the logarithmic fields in the affine Jordan cells are quasi-primary, and discuss the conformal and $SL(2, \mathbb{R})$ Ward identities which follow. Without making any simplifying assumptions about the operator-product expansions of the fields, we find the general solutions for two- and three-point chiral blocks. Our results thus cover all the possible cases based on primary fields not belonging to affine Jordan cells, primary fields belonging to affine Jordan cells, and the logarithmic partner fields completing the affine Jordan cells.

Most of our computations are based on the introduction of generating functions for the fields appearing in the various representations. This means that the affine correlators of the individual fields are obtained by expanding certain generating-function chiral blocks.

A modification of the Knizhnik-Zamolodchikov (KZ) equations [10] is required to cover affine Jordan cells in addition to primary fields. It is demonstrated that the chiral blocks obtained as solutions to the Ward identities satisfy these generalized KZ equations. This verification is straightforward once it has been established that the two- and three-point chiral blocks may be expressed compactly in terms of spins with nilpotent parts. We show that this is possible. It also follows that a two- or three-point chiral block factorizes into a ‘conformal’ part and a ‘group’ part.

The chiral blocks are found to exhibit hierarchical structures obtained by computing derivatives with respect to the spins. This extends an observation made in [11, 12, 13] that the sets of two- and three-point conformal blocks in logarithmic CFT are linked via derivatives with respect to the conformal weights.

It is noted that a simple distinction has been introduced as we refer to chiral correlators in the WZW model as *chiral blocks*, while chiral correlators in logarithmic CFT are referred to as *conformal blocks*. This is common practice.

A merit of our construction seems to be that the affine correlators reduce to the conformal ones when a straightforward extension of the prescription for hamiltonian reduction introduced in [14, 15] is employed. The idea is formulated in the realm of generating functions for the Virasoro primary fields in spin- j multiplets, and we find that it may be extended to affine Jordan cells and thus cover the reduction of our logarithmic $SL(2, \mathbb{R})$ WZW model to logarithmic CFT.

This paper proceeds as follows. To fix our notation and to prepare for the discussion of hamiltonian reduction, we first review the recently obtained general solutions to the conformal Ward identities for two- and three-point conformal blocks in logarithmic CFT [13]. This is followed by a discussion in Section 3 of generating-function primary fields and their correlators in $SL(2, \mathbb{R})$ WZW models. This is the framework which we extend in Section 4 and eventually use in our analysis of chiral blocks. In Section 4, we thus describe the indecomposable $sl(2)$ representations underlying the affine Jordan cells. The correspondingly modified KZ equations are also introduced. Section 5 concerns the explicit results on two- and three-point chiral blocks. It includes a discussion of the factorization of the chiral blocks based on spins with nilpotent parts, as well as a discussion of the hierarchical structures of the chiral blocks.

Some technical details are deferred to Appendix A. The extended prescription for hamiltonian reduction is considered in Section 6, where it is demonstrated that our chiral blocks reduce to the conformal ones reviewed in Section 2. Section 7 contains some concluding remarks.

2 Correlators in logarithmic CFT

2.1 Conformal Jordan cell

A conformal Jordan cell of rank two consists of two fields: a primary field, Φ , of conformal weight $\hat{\Delta}$ and its non-primary, ‘logarithmic’ partner field, Ψ , on which the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{\hat{c}}{12}n(n^2 - 1)\delta_{n+m,0} \quad (1)$$

generated by $\{L_n\}$ does not act diagonally. The central extension is denoted \hat{c} . With a conventional relative normalization of the fields, we have

$$\begin{aligned} [L_n, \Phi(z)] &= \left(z^{n+1}\partial_z + \hat{\Delta}(n+1)z^n \right) \Phi(z) \\ [L_n, \Psi(z)] &= \left(z^{n+1}\partial_z + \hat{\Delta}(n+1)z^n \right) \Psi(z) + (n+1)z^n \Phi(z) \end{aligned} \quad (2)$$

It has been suggested by Flohr [16] to describe these fields in a unified way by introducing a nilpotent, yet even, parameter $\hat{\theta}$ satisfying $\hat{\theta}^2 = 0$. We will follow this idea here, though use an approach closer to the one employed in [17, 18, 13]. We thus define the field or unified cell

$$\Upsilon(z; \hat{\theta}) = \Phi(z) + \hat{\theta}\Psi(z) \quad (3)$$

which is seen to be ‘primary’ of conformal weight $\hat{\Delta} + \hat{\theta}$ as the commutators (2) may be expressed as

$$[L_n, \Upsilon(z; \hat{\theta})] = \left(z^{n+1}\partial_z + (\hat{\Delta} + \hat{\theta})(n+1)z^n \right) \Upsilon(z; \hat{\theta}) \quad (4)$$

Following [13], a primary field belonging to a conformal Jordan cell is referred to as a cellular primary field. A primary field *not* belonging to a conformal Jordan cell may be represented as $\Upsilon(z; 0)$, and we will reserve this notation for these non-cellular primary fields. To avoid ambiguities, we will therefore refrain from considering unified cells $\Upsilon(z; \hat{\theta})$, as defined in (3), for vanishing $\hat{\theta}$.

2.2 Conformal Ward identities

We consider *quasi-primary fields* only, ensuring the projective invariance of their correlators constructed by sandwiching the fields between projectively invariant vacua. This invariance is made manifest qua the conformal Ward identities which are given here for \mathcal{N} -point conformal blocks:

$$\begin{aligned} 0 &= \sum_{i=1}^{\mathcal{N}} \partial_{z_i} \langle \Upsilon_1(z_1; \hat{\theta}_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}; \hat{\theta}_{\mathcal{N}}) \rangle \\ 0 &= \sum_{i=1}^{\mathcal{N}} \left(z_i \partial_{z_i} + \hat{\Delta}_i + \hat{\theta}_i \right) \langle \Upsilon_1(z_1; \hat{\theta}_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}; \hat{\theta}_{\mathcal{N}}) \rangle \\ 0 &= \left(\mathcal{L}^{\mathcal{N}} + 2 \sum_{i=1}^{\mathcal{N}} \hat{\theta}_i z_i \right) \langle \Upsilon_1(z_1; \hat{\theta}_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}; \hat{\theta}_{\mathcal{N}}) \rangle \end{aligned} \quad (5)$$

To simplify the notation, we have introduced the differential operator

$$\mathcal{L}^{\mathcal{N}} = \sum_{i=1}^{\mathcal{N}} \left(z_i^2 \partial_{z_i} + 2\hat{\Delta}_i z_i \right) \quad (6)$$

Information on the individual correlators may be extracted from solutions to the conformal Ward identities involving unified cells. In the case of

$$\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; \hat{\theta}_3) \rangle \quad (7)$$

for example, the third conformal Ward identity (5) reads

$$0 = \left(\mathcal{L}^3 + 2(\hat{\theta}_1 z_1 + \hat{\theta}_3 z_3) \right) \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; \hat{\theta}_3) \rangle \quad (8)$$

A solution to the complete set of conformal Ward identities is an expression expandable in $\hat{\theta}_1$ and $\hat{\theta}_3$. The term proportional to $\hat{\theta}_1$ but independent of $\hat{\theta}_3$, for example, should then be identified with $\langle \Psi_1(z_1) \Upsilon_2(z_2; 0) \Phi_3(z_3) \rangle$.

By construction, and as illustrated by this example, correlators involving unified cells and non-cellular primary fields may thus be regarded as generating-function correlators whose expansions in the nilpotent parameters give the individual correlators involving combinations of cellular primary fields, non-cellular primary fields, and logarithmic fields. Our focus will therefore be on correlators of combinations of unified cells and non-cellular primary fields.

2.3 Two-point conformal blocks

Based on the ansatz

$$\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \rangle = \frac{\hat{A}(\hat{\theta}_1, \hat{\theta}_2) + \hat{B}(\hat{\theta}_1, \hat{\theta}_2) \ln z_{12}}{z_{12}^{2h}} \quad (9)$$

where

$$\hat{A}(\hat{\theta}_1, \hat{\theta}_2) = \hat{A}^0 + \hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 \quad (10)$$

and similarly for $\hat{B}(\hat{\theta}_1, \hat{\theta}_2)$, the general (generating-function) two-point conformal blocks read

$$\begin{aligned} \langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; 0) \rangle &= \hat{A}^0 V_2 \\ \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \rangle &= \hat{A}^1 \hat{\theta}_1 V_2 \\ \langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; \hat{\theta}_2) \rangle &= \hat{A}^2 \hat{\theta}_2 V_2 \\ \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \rangle &= \left\{ \hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \left(\hat{A}^{12} - 2\hat{A}^1 \ln z_{12} \right) \hat{\theta}_1 \hat{\theta}_2 \right\} V_2 \end{aligned} \quad (11)$$

Here we have introduced the shorthand notation

$$V_2 = \frac{\delta_{\hat{\Delta}_1, \hat{\Delta}_2}}{z_{12}^{\hat{\Delta}_1 + \hat{\Delta}_2}} \quad (12)$$

To keep the notation simple, we are using the standard abbreviation $z_{ij} = z_i - z_j$. It is understood that an \hat{A}^1 , for example, appearing in one (generating-function) correlator a priori is independent of an \hat{A}^1 appearing in another. Also, even though \hat{A}^2 does not appear explicitly in some of these expressions, it may nevertheless be related to \hat{A}^1 . For the sake of simplicity, the solutions listed here are merely indicating the general form and the degrees of freedom without reference to the fate of all the various parameters appearing in the ansatz (9). Similar comments also apply to the results on correlators discussed in the following. Finally, the solutions for the individual two-point conformal blocks are easily extracted [13].

By considering $\hat{\theta}_i$ as the nilpotent part of the generalized conformal weight $\hat{\Delta}_i + \hat{\theta}_i$ [17, 13], one may represent the results (11) as

$$\begin{aligned}
\langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; 0) \rangle &= \delta_{\hat{\Delta}_1, \hat{\Delta}_2} \frac{\hat{A}^0}{z_{12}^{\hat{\Delta}_1 + \hat{\Delta}_2}} \\
\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \rangle &= \delta_{\hat{\Delta}_1, \hat{\Delta}_2} \frac{\hat{A}^1 \hat{\theta}_1}{z_{12}^{(\hat{\Delta}_1 + \hat{\theta}_1) + \hat{\Delta}_2}} \\
\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \rangle &= \delta_{\hat{\Delta}_1, \hat{\Delta}_2} \frac{\hat{A}^1 \hat{\theta}_1 + \hat{A}^1 \hat{\theta}_2 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2}{z_{12}^{(\hat{\Delta}_1 + \hat{\theta}_1) + (\hat{\Delta}_2 + \hat{\theta}_2)}} \quad (13)
\end{aligned}$$

The similar expression for the correlator $\langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; \hat{\theta}_2) \rangle$ is obtained from the second one by interchanging the indices.

2.4 Three-point conformal blocks

Based on the ansatz

$$\begin{aligned}
&\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \Upsilon_3(z_3; \hat{\theta}_3) \rangle \\
&= \left\{ \hat{A}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) + \hat{B}_1(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{12} + \hat{B}_2(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{23} + \hat{B}_3(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{13} \right. \\
&\quad + \hat{D}_{11}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln^2 z_{12} + \hat{D}_{12}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{12} \ln z_{23} + \hat{D}_{13}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{12} \ln z_{13} \\
&\quad \left. + \hat{D}_{22}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln^2 z_{23} + \hat{D}_{23}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln z_{23} \ln z_{13} + \hat{D}_{33}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \ln^2 z_{13} \right\} z_{12}^{-h_1} z_{23}^{-h_2} z_{13}^{-h_3} \quad (14)
\end{aligned}$$

where

$$\hat{A}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) = \hat{A}^0 + \hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \hat{A}^3 \hat{\theta}_3 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 + \hat{A}^{23} \hat{\theta}_2 \hat{\theta}_3 + \hat{A}^{13} \hat{\theta}_1 \hat{\theta}_3 + \hat{A}^{123} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \quad (15)$$

and similarly for $\hat{B}_i(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and $\hat{D}_{ij}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$, the general (generating-function) three-point conformal blocks read

$$\begin{aligned}
&\langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle = \hat{A}^0 V_3 \\
&\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle = \left\{ \hat{A}^0 + \hat{A}^1 \hat{\theta}_1 - \hat{A}^0 \hat{\theta}_1 \ln \frac{z_{12} z_{13}}{z_{23}} \right\} V_3 \\
&\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \Upsilon_3(z_3; 0) \rangle \\
&= \left\{ \hat{A}^0 + \hat{A}^1 \hat{\theta}_1 - \hat{A}^0 \hat{\theta}_1 \ln \frac{z_{12} z_{13}}{z_{23}} + \hat{A}^2 \hat{\theta}_2 - \hat{A}^0 \hat{\theta}_2 \ln \frac{z_{12} z_{23}}{z_{13}} \right. \\
&\quad \left. + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 - \hat{A}^1 \hat{\theta}_1 \hat{\theta}_2 \ln \frac{z_{12} z_{23}}{z_{13}} - \hat{A}^2 \hat{\theta}_1 \hat{\theta}_2 \ln \frac{z_{12} z_{13}}{z_{23}} + \hat{A}^0 \hat{\theta}_1 \hat{\theta}_2 \ln \frac{z_{12} z_{23}}{z_{13}} \ln \frac{z_{12} z_{13}}{z_{23}} \right\} V_3 \\
&\langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \Upsilon_3(z_3; \hat{\theta}_3) \rangle \\
&= \left\{ \hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \hat{A}^3 \hat{\theta}_3 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 - \hat{A}^1 \hat{\theta}_1 \hat{\theta}_2 \ln \frac{z_{12} z_{23}}{z_{13}} - \hat{A}^2 \hat{\theta}_1 \hat{\theta}_2 \ln \frac{z_{12} z_{13}}{z_{23}} \right. \\
&\quad + \hat{A}^{23} \hat{\theta}_2 \hat{\theta}_3 - \hat{A}^2 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{23} z_{13}}{z_{12}} - \hat{A}^3 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{13}}{z_{12} z_{23}} + \hat{A}^{13} \hat{\theta}_1 \hat{\theta}_3 - \hat{A}^1 \hat{\theta}_1 \hat{\theta}_3 \ln \frac{z_{23} z_{13}}{z_{12}} \\
&\quad - \hat{A}^3 \hat{\theta}_1 \hat{\theta}_3 \ln \frac{z_{12} z_{13}}{z_{23}} + \hat{A}^{123} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 - \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{23} z_{13}}{z_{12}} \\
&\quad \left. - \hat{A}^{23} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{12} z_{13}}{z_{23}} - \hat{A}^{13} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{12} z_{23}}{z_{13}} + \hat{A}^1 \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{23} z_{12}}{z_{13}} \ln \frac{z_{23} z_{13}}{z_{12}} \right\}
\end{aligned}$$

$$+ \hat{A}^2 \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{12} z_{13}}{z_{23}} \ln \frac{z_{23} z_{13}}{z_{12}} + \hat{A}^3 \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3 \ln \frac{z_{12} z_{23}}{z_{13}} \ln \frac{z_{12} z_{13}}{z_{23}} \Big\} V_3 \quad (16)$$

Here we have introduced the abbreviation

$$V_3 = \frac{1}{z_{12}^{\hat{\Delta}_1 + \hat{\Delta}_2 - \hat{\Delta}_3} z_{23}^{-\hat{\Delta}_1 + \hat{\Delta}_2 + \hat{\Delta}_3} z_{13}^{\hat{\Delta}_1 - \hat{\Delta}_2 + \hat{\Delta}_3}} \quad (17)$$

The remaining correlators are obtained by appropriate permutations in the indices.

As in the case of two-point conformal blocks, the three-point conformal blocks may be represented in terms of generalized conformal weights, $\hat{\Delta}_i + \hat{\theta}_i$:

$$\begin{aligned} \langle \Upsilon_1(z_1; 0) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle &= \frac{\hat{A}^0}{z_{12}^{\hat{\Delta}_1 + \hat{\Delta}_2 - \hat{\Delta}_3} z_{23}^{-\hat{\Delta}_1 + \hat{\Delta}_2 + \hat{\Delta}_3} z_{13}^{\hat{\Delta}_1 - \hat{\Delta}_2 + \hat{\Delta}_3}} \\ \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle &= \frac{\hat{A}^0 + \hat{A}^1 \hat{\theta}_1}{z_{12}^{(\hat{\Delta}_1 + \hat{\theta}_1) + \hat{\Delta}_2 - \hat{\Delta}_3} z_{23}^{-(\hat{\Delta}_1 + \hat{\theta}_1) + \hat{\Delta}_2 + \hat{\Delta}_3} z_{13}^{(\hat{\Delta}_1 + \hat{\theta}_1) - \hat{\Delta}_2 + \hat{\Delta}_3}} \\ \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \Upsilon_3(z_3; 0) \rangle &= \frac{\hat{A}^0 + \hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2}{z_{12}^{(\hat{\Delta}_1 + \hat{\theta}_1) + (\hat{\Delta}_2 + \hat{\theta}_2) - \hat{\Delta}_3} z_{23}^{-(\hat{\Delta}_1 + \hat{\theta}_1) + (\hat{\Delta}_2 + \hat{\theta}_2) + \hat{\Delta}_3} z_{13}^{(\hat{\Delta}_1 + \hat{\theta}_1) - (\hat{\Delta}_2 + \hat{\theta}_2) + \hat{\Delta}_3}} \\ \langle \Upsilon_1(z_1; \hat{\theta}_1) \Upsilon_2(z_2; \hat{\theta}_2) \Upsilon_3(z_3; \hat{\theta}_3) \rangle &= \frac{\hat{A}^1 \hat{\theta}_1 + \hat{A}^2 \hat{\theta}_2 + \hat{A}^3 \hat{\theta}_3 + \hat{A}^{12} \hat{\theta}_1 \hat{\theta}_2 + \hat{A}^{23} \hat{\theta}_2 \hat{\theta}_3 + \hat{A}^{13} \hat{\theta}_1 \hat{\theta}_3 + \hat{A}^{123} \hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3}{z_{12}^{(\hat{\Delta}_1 + \hat{\theta}_1) + (\hat{\Delta}_2 + \hat{\theta}_2) - (\hat{\Delta}_3 + \hat{\theta}_3)} z_{23}^{-(\hat{\Delta}_1 + \hat{\theta}_1) + (\hat{\Delta}_2 + \hat{\theta}_2) + (\hat{\Delta}_3 + \hat{\theta}_3)} z_{13}^{(\hat{\Delta}_1 + \hat{\theta}_1) - (\hat{\Delta}_2 + \hat{\theta}_2) + (\hat{\Delta}_3 + \hat{\theta}_3)}} \end{aligned} \quad (18)$$

The remaining four combinations are obtained by appropriate permutations in the indices.

2.5 Hierarchical structures for conformal blocks

Based on ideas discussed in [12, 11], it was found in [13] that the correlators involving logarithmic fields may be represented as follows:

$$\begin{aligned} \langle \Psi_1(z_1) \Upsilon_2(z_2; 0) \rangle &= \hat{A}^1 V_2 \\ \langle \Psi_1(z_1) \Phi_2(z_2) \rangle &= \hat{A}^1 V_2 \\ \langle \Psi_1(z_1) \Psi_2(z_2) \rangle &= \left(\hat{A}^{12} + \hat{A}^2 \partial_{\hat{\Delta}_1} + \hat{A}^1 \partial_{\hat{\Delta}_2} \right) V_2 \\ \langle \Psi_1(z_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle &= \left(\hat{A}^1 + \hat{A}^0 \partial_{\hat{\Delta}_1} \right) V_3 \\ \langle \Psi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3; 0) \rangle &= \left(\hat{A}^1 + \hat{A}^0 \partial_{\hat{\Delta}_1} \right) V_3 \\ \langle \Psi_1(z_1) \Psi_2(z_2) \Upsilon_3(z_3; 0) \rangle &= \left(\hat{A}^{12} + \hat{A}^1 \partial_{\hat{\Delta}_2} + \hat{A}^2 \partial_{\hat{\Delta}_1} + \hat{A}^0 \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \right) V_3 \\ \langle \Psi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle &= \hat{A}^1 V_3 \\ \langle \Psi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle &= \left(\hat{A}^{12} + \hat{A}^2 \partial_{\hat{\Delta}_1} + \hat{A}^1 \partial_{\hat{\Delta}_2} \right) V_3 \\ \langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle &= \left(\hat{A}^{123} + \hat{A}^{23} \partial_{\hat{\Delta}_1} + \hat{A}^{13} \partial_{\hat{\Delta}_2} + \hat{A}^{12} \partial_{\hat{\Delta}_3} \right. \\ &\quad \left. + \hat{A}^3 \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} + \hat{A}^1 \partial_{\hat{\Delta}_2} \partial_{\hat{\Delta}_3} + \hat{A}^2 \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_3} \right) V_3 \end{aligned} \quad (19)$$

in addition to expressions obtained by appropriately permuting the indices. One may therefore represent the correlators hierarchically as

$$\langle \Psi_1(z_1) \Upsilon_2(z_2; 0) \rangle = \hat{A}^1 V_2 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Upsilon_2(z_2; 0) \rangle$$

$$\begin{aligned}
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle &= \hat{A}^1 V_2 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Phi_2(z_2) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle &= \hat{A}^{12} V_2 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Psi_2(z_2) \rangle + \partial_{\hat{\Delta}_2} \langle \Psi_1(z_1) \Phi_2(z_2) \rangle - \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \langle \Phi_1(z_1) \Phi_2(z_2) \rangle
\end{aligned} \tag{20}$$

in the case of two-point conformal blocks, and

$$\begin{aligned}
\langle \Psi_1(z_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle &= \hat{A}^1 V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Upsilon_2(z_2; 0) \Upsilon_3(z_3; 0) \rangle \\
\langle \Psi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3; 0) \rangle &= \hat{A}^1 V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3; 0) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) \Upsilon_3(z_3; 0) \rangle &= \hat{A}^{12} V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Psi_2(z_2) \Upsilon_3(z_3; 0) \rangle + \partial_{\hat{\Delta}_2} \langle \Psi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3; 0) \rangle \\
&\quad - \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \langle \Phi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3; 0) \rangle \\
\langle \Psi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle &= \hat{A}^1 V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle &= \hat{A}^{12} V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle + \partial_{\hat{\Delta}_2} \langle \Psi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle \\
&\quad - \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle &= \hat{A}^{123} V_3 + \partial_{\hat{\Delta}_1} \langle \Phi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle + \partial_{\hat{\Delta}_2} \langle \Psi_1(z_1) \Phi_2(z_2) \Psi_3(z_3) \rangle \\
&\quad + \partial_{\hat{\Delta}_3} \langle \Psi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle - \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \langle \Phi_1(z_1) \Phi_2(z_2) \Psi_3(z_3) \rangle \\
&\quad - \partial_{\hat{\Delta}_2} \partial_{\hat{\Delta}_3} \langle \Psi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle - \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_3} \langle \Phi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle \\
&\quad + \partial_{\hat{\Delta}_1} \partial_{\hat{\Delta}_2} \partial_{\hat{\Delta}_3} \langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle
\end{aligned} \tag{21}$$

in the case of three-point conformal blocks. As above, the remaining correlators may be obtained by appropriately permuting the indices.

3 On $SL(2, \mathbb{R})$ WZW models

The affine $sl(2)_k$ Lie algebra, including the commutators with the Virasoro modes, reads

$$\begin{aligned}
[J_{+,n}, J_{-,m}] &= 2J_{0,n+m} + kn\delta_{n+m,0} \\
[J_{0,n}, J_{\pm,m}] &= \pm J_{\pm,n+m} \\
[J_{0,n}, J_{0,m}] &= \frac{k}{2}n\delta_{n+m,0} \\
[L_n, J_{a,m}] &= -mJ_{a,n+m}
\end{aligned} \tag{22}$$

Another conventional notation is obtained by replacing $\{J_{+,n}, 2J_{0,n}, J_{-,n}\}$ by $\{E_n, H_n, F_n\}$. The level of the algebra is indicated by k and is related to the central charge as $c = 3k/(k+2)$. The non-vanishing entries of the Cartan-Killing form of $sl(2)$ are given by

$$\kappa_{00} = \frac{1}{2}, \quad \kappa_{+-} = \kappa_{-+} = 1 \tag{23}$$

and appear as coefficients to the central terms in (22). Its inverse is given by

$$\kappa^{00} = 2, \quad \kappa^{+-} = \kappa^{-+} = 1 \tag{24}$$

and comes into play when discussing the affine Sugawara construction below. We will be concerned mainly with the ‘horizontal’ part of the affine Lie algebra, the $sl(2)$ Lie algebra generated by the zero modes $\{J_a := J_{a,0}\}$.

We will assume that the Virasoro primary fields of a given conformal weight Δ may be organized in multiplets corresponding to spin- j representations of the $sl(2)$ algebra, where

$$\Delta = \frac{j(j+1)}{k+2} \tag{25}$$

In the following, j is taken to be real even though the general formalism is amenable to treat j complex as well. If $2j$ is a non-negative integer, we may label the $2j + 1$ members of the associated multiplet as in

$$\phi_{-j}(z), \phi_{-j+1}(z), \dots, \phi_{j-1}(z), \phi_j(z) \quad (26)$$

where the dependence on j , often indicated by $\phi_{j,m}$, is suppressed. A finite-dimensional representation like (26) is often referred to as an integrable representation. The field ϕ_m has J_0 eigenvalue m , while we will use the following convenient choice of relative normalizations of the fields:

$$\begin{aligned} [J_+, \phi_m(z)] &= (j + m + 1)\phi_{m+1}(z) \\ [J_0, \phi_m(z)] &= m\phi_m(z) \\ [J_-, \phi_m(z)] &= (j - m + 1)\phi_{m-1}(z) \end{aligned} \quad (27)$$

If $2j$ is not a non-negative integer, the associated primary fields may in general be organized in an infinite-dimensional multiplet corresponding to an $sl(2)$ representation.

3.1 Generating-function primary fields

A generating function for the $2j + 1$ Virasoro primary fields in an integrable representation may be written [19]

$$\phi(z, x) = \sum_{m=-j}^j \phi_m(z) x^{j-m} \quad (28)$$

To keep the notation simple here and in the following, we do not indicate explicitly whether the sum is over integers or half-integers as this should be obvious from the integer or half-integer nature of the spin itself. For general spin and associated infinite-dimensional multiplet, the generating function for a so-called highest-weight representation, for example, reads

$$\phi(z, x) = \sum_{m \in (j + \mathbb{Z}_{\leq})} \phi_m(z) x^{j-m} \quad (29)$$

The adjoint action of the affine generators on the generating-function primary field reads

$$[J_{a,n}, \phi(z, x)] = -z^n D_a(x) \phi(z, x) \quad (30)$$

where the differential operators $D_a(x)$ are defined by

$$\begin{aligned} D_+(x) &= x^2 \partial_x - 2jx \\ D_0(x) &= x \partial_x - j \\ D_-(x) &= -\partial_x \end{aligned} \quad (31)$$

They generate the Lie algebra $sl(2)$, and one recovers (27) from (30).

A correlator like the \mathcal{N} -point chiral block

$$\langle \phi_1(z_1, x_1) \dots \phi_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}) \rangle \quad (32)$$

is seen to correspond to a generating function for the individual correlators based on fields, $\phi_{i,m_i}(z_i)$, appearing in expansions like (28) (or (29), for example). That is, the \mathcal{N} -point chiral block

$$\langle \phi_{1,m_1}(z_1) \dots \phi_{\mathcal{N},m_{\mathcal{N}}}(z_{\mathcal{N}}) \rangle \quad (33)$$

appears as the coefficient to $\prod_{i=1}^{\mathcal{N}} x_i^{j_i - m_i}$ in an expansion of (32). The general expansion thus reads

$$\langle \phi_1(z_1, x_1) \dots \phi_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}) \rangle = \sum_{m_1, \dots, m_{\mathcal{N}}} \langle \phi_{1,m_1}(z_1) \dots \phi_{\mathcal{N},m_{\mathcal{N}}}(z_{\mathcal{N}}) \rangle x_1^{j_1 - m_1} \dots x_{\mathcal{N}}^{j_{\mathcal{N}} - m_{\mathcal{N}}} \quad (34)$$

where the ranges of the summation variables depend on the individual spin- j_i representations.

3.2 The KZ equations

In a WZW model, the Virasoro generators are realized as bilinear expressions in the affine generators. This is referred to as the affine Sugawara construction which is here written in terms of modes

$$L_N = \frac{1}{2(k+2)} \kappa^{ab} \left(\sum_{n \leq -1} J_{a,n} J_{b,N-n} + \sum_{n \geq 0} J_{a,N-n} J_{b,n} \right) \quad (35)$$

Here and in the following, we will use the convention of summing over appropriately repeated group indices, $a = \pm, 0$. Acting on a highest-weight state, the affine Sugawara construction gives rise to singular vectors of the combined algebra. The decoupling of these is trivial for $N > 0$. For $N = 0$, it reproduces the relation (25) as the eigenvalue of L_0 is equated with the eigenvalue of the normalized quadratic Casimir:

$$\Delta = \frac{\kappa^{ab} D_a(x) D_b(x)}{2(k+2)} = \frac{j(j+1)}{k+2} \quad (36)$$

The condition corresponding to $N = -1$ leads to the celebrated KZ equations [10] which are written here for an \mathcal{N} -point chiral block of generating-function primary fields

$$0 = K Z_i \langle \phi_1(z_1, x_1) \dots \phi_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}) \rangle, \quad i = 1, \dots, \mathcal{N} \quad (37)$$

where

$$K Z_i = \left((k+2) \partial_{z_i} - \sum_{j \neq i} \frac{\kappa^{ab} D_a(x_i) D_b(x_j)}{z_i - z_j} \right) \quad (38)$$

These \mathcal{N} differential equations associated to a given \mathcal{N} -point chiral block are not all independent. This is easily illustrated by considering the sum

$$\sum_{i=1}^{\mathcal{N}} K Z_i = (k+2) \sum_{i=1}^{\mathcal{N}} \partial_{z_i} \quad (39)$$

which merely induces translational invariance already imposed by the first conformal Ward identity (5). As we will discuss below, a simple modification of the KZ equations (37), (38) apply to correlators involving certain logarithmic fields to be introduced in the following.

4 Affine Jordan cells

We wish to consider the situation where every Virasoro primary field in a given $sl(2)$ representation may have a logarithmic partner. The resulting multiplet of fields is comprised of primary fields as well as so-called logarithmic fields and will be referred to as an affine Jordan cell. A priori, the hosting model may consist of a family of affine Jordan cells in coexistence with an independent family of multiplets of primary fields without logarithmic partners. We will refer loosely to such a model as a logarithmic WZW model. Primary fields not appearing in an affine Jordan cell will be called non-cellular primary fields. It is found that the affine Jordan cells relevant to our studies contain primary fields not having logarithmic partners. These primary fields are naturally included in the generating functions for the logarithmic fields rather than in the generating functions for the primary fields comprising the original spin- j representation we are extending. To reach this appreciation of the affine Jordan cells, we initially consider an extension of the differential-operator realization (31) and its role in a generalization of (30).

4.1 Generating-function unified cells

The differential-operator realization

$$\begin{aligned} D_+(x; \theta) &= x^2 \partial_x - 2(j + \theta)x \\ D_0(x; \theta) &= x \partial_x - (j + \theta) \\ D_-(x; \theta) &= -\partial_x \end{aligned} \quad (40)$$

of the Lie algebra $sl(2)$ is designed to act on a representation of (generalized) spin $j + \theta$, where θ is a nilpotent, yet even, parameter satisfying $\theta^2 = 0$. Extending the idea of organizing fields in generating functions as in (28) satisfying (30), we introduce the formal generating-function unified cell $\Upsilon(z, x; \theta)$ satisfying

$$[J_a, \Upsilon(z, x; \theta)] = -D_a(x; \theta) \Upsilon(z, x; \theta) \quad (41)$$

We note that this also applies to generating-function primary fields as it reduces to (30) (for $n = 0$) when we set $\theta = 0$. Here and in the following, focus is on the $sl(2)$ Lie algebra part of the affine generators. An expansion of the generating-function unified cell with respect to θ may be written

$$\Upsilon(z, x; \theta) = \Phi(z, x) + \theta \Psi(z, x) \quad (42)$$

resembling the definition of the unified cell (3) in logarithmic CFT. In terms of the new generating functions, $\Phi(z, x)$ and $\Psi(z, x)$, the commutators (41) read

$$\begin{aligned} [J_+, \Phi(z, x)] &= -D_+(x) \Phi(z, x) \\ [J_+, \Psi(z, x)] &= -D_+(x) \Psi(z, x) + 2x \Phi(z, x) \\ [J_0, \Phi(z, x)] &= -D_0(x) \Phi(z, x) \\ [J_0, \Psi(z, x)] &= -D_0(x) \Psi(z, x) + \Phi(z, x) \\ [J_-, \Phi(z, x)] &= -D_-(x) \Phi(z, x) \\ [J_-, \Psi(z, x)] &= -D_-(x) \Psi(z, x) \end{aligned} \quad (43)$$

where the differential operators, $D_a(x)$, are given in (31).

These commutators severely restrict the set of $sl(2)$ representations for which the two fields $\Phi(z, x)$ and $\Psi(z, x)$ can be considered generating functions. It is beyond the scope of the present work, though, to classify these representations, even in the simple case where $\Phi(z, x)$ is the generating function for a finite-dimensional representation as in (28). We hope to address this classification elsewhere. Here we merely wish to demonstrate the existence of representations corresponding to the generating functions (42), (43) and to illustrate their complexity. We will do so by considering a particular logarithmic extension of a finite-dimensional spin- j representation. More general examples are considered in Section 4.2.

We thus introduce the following expansions of the generating functions $\Phi(z, x)$ and $\Psi(z, x)$:

$$\Phi(z, x) = \sum_{m=-j}^j \Phi_m(z) x^{j-m}, \quad \Psi(z, x) = \sum_{m=-\infty}^j \Psi_m(z) x^{j-m} \quad (44)$$

The remark following (28) about m taking on integer or half-integer values also applies when one (or even both) of the summation bounds is (either plus or minus) infinity. As already mentioned, we are concerned with Jordan cells whose principal parts correspond to finite-dimensional spin- j representations, here governed by the generating function $\Phi(z, x)$. It is noted that the logarithmic part, on the other hand, consists of infinitely many fields.

With the understanding that $\Phi_m(z)$ only exists for $m = -j, \dots, j$ while $\Psi_m(z)$ only exists for $m = -\infty, \dots, j$, the adjoint action of the $sl(2)$ Lie algebra on the modes of the two generating functions (44)

may be written compactly as

$$\begin{aligned}
[J_+, \Phi_m(z)] &= (j + m + 1)\Phi_{m+1}(z) \\
[J_+, \Psi_m(z)] &= (j + m + 1)\Psi_{m+1}(z) + 2\Phi_{m+1}(z) \\
[J_0, \Phi_m(z)] &= m\Phi_m(z) \\
[J_0, \Psi_m(z)] &= m\Psi_m(z) + \Phi_m(z) \\
[J_-, \Phi_m(z)] &= (j - m + 1)\Phi_{m-1}(z) \\
[J_-, \Psi_m(z)] &= (j - m + 1)\Psi_{m-1}(z)
\end{aligned} \tag{45}$$

This is equivalent to simply setting a non-existing field equal to zero whenever it formally appears in (45). The following diagram may help visualizing the representation:

$$\begin{array}{cccccccccccc}
\ldots & \xleftrightarrow{J_-, J_+} & \Psi_{-j-1} & \xleftarrow{J_-} & \Psi_{-j} & \xleftrightarrow{J_-, J_+} & \Psi_{-j+1} & \xleftrightarrow{J_-, J_+} & \ldots & \xleftrightarrow{J_-, J_+} & \Psi_j \\
& & & & \searrow J_+ & \downarrow J_0 & \searrow J_+ & \downarrow J_0 & \searrow J_+ & \ldots & \searrow J_+ & \downarrow J_0 \\
& & & & \Phi_{-j} & \xleftrightarrow{J_-, J_+} & \Phi_{-j+1} & \xleftrightarrow{J_-, J_+} & \ldots & \xleftrightarrow{J_-, J_+} & \Phi_j
\end{array} \tag{46}$$

Here the arrows indicate the adjoint actions of J_a (except the primary parts of the adjoint actions of J_0 which are not indicated explicitly). It is observed that only $2j + 1$ of the fields $\Psi_m(z)$ are logarithmic fields, where a logarithmic field is characterized by the property that at least one of the affine generators acts non-diagonally on them. Here, in particular, they do not have well-defined J_0 eigenvalues.

The naive expansion where $\Psi(z, x)$ is a sum of $2j + 1$ fields similar to the expansion of $\Phi(z, x)$ turns out to be inconsistent. The same problem occurs when trying to write $\Psi(z, x)$ as an infinite sum from $-j$ to ∞ , as it actually occurs for all power-series expansions of $\Psi(z, x)$ having lowest magnetic moment, m , equal to $-j$. This asymmetry in extensions beyond j and $-j$, respectively, stems from the fact that J_+ may act non-diagonally (in the algebraic sense (45), i.e., diagonally in the diagram (46)) while J_- only acts diagonally (i.e., horizontally in the diagram (46)). One can extend in both directions simultaneously

$$\Phi(z, x) = \sum_{m=-j}^j \Phi_m(z) x^{j-m}, \quad \Psi(z, x) = \sum_{m=-\infty}^{\infty} \Psi_m(z) x^{j-m} \tag{47}$$

where $\Upsilon_i(z_i, x_i; \theta_i)$ denotes a generating-function primary field if $\theta_i = 0$. As above, the ranges of the summation variables depend on the individual $sl(2)$ representations.

4.2 More on indecomposable $sl(2)$ representations

It is stressed that the differential-operator realization (40) and the generating-function (49) do not exhaust all possible extensions of the ordinary WZW model outlined in Section 3. This is illustrated by the models discussed in [6, 7, 8, 4, 9], for example, and will be addressed further elsewhere. The construction developed in the present work has the virtue that it, under hamiltonian reduction, reduces to the non-affine logarithmic CFT reviewed in Section 2. This will be the topic of Section 6 below. Here we wish to indicate the level of complexity of the $sl(2)$ representations associated to more general expansions of $\Psi(z, x)$ than power-series expansions such as (47).

To this end, we consider the expansions

$$\Phi(z, x) = \sum_m \Phi_m(z) x^{j-m}, \quad \Psi(z, x) = \sum_{m,n} \Psi_{m,n}(z) x^{j-m} \ln^n x \quad (52)$$

where we have left the summation ranges unspecified. In terms of these modes, the commutators (43) read

$$\begin{aligned} [J_+, \Phi_m(z)] &= (j + m + 1) \Phi_{m+1}(z) \\ [J_+, \Psi_{m,n}(z)] &= (j + m + 1) \Psi_{m+1,n}(z) - (n + 1) \Psi_{m+1,n+1}(z) + 2 \Phi_{m+1}(z) \\ [J_0, \Phi_m(z)] &= m \Phi_m(z) \\ [J_0, \Psi_{m,n}(z)] &= m \Psi_{m+1,n}(z) - (n + 1) \Psi_{m,n+1}(z) + \Phi_m(z) \\ [J_-, \Phi_m(z)] &= (j - m + 1) \Phi_{m-1}(z) \\ [J_-, \Psi_{m,n}(z)] &= (j - m + 1) \Psi_{m-1,n}(z) + (n + 1) \Psi_{m-1,n+1}(z) \end{aligned} \quad (53)$$

where a field whose indices do not match the expansion (52) is set to zero. To illustrate such an indecomposable $sl(2)$ representation, we let n run from 0 to 2 and focus on a typical sequence in the magnetic moments: $m - 1, m, m + 1$, where $-j < m < j$. The announced part of the corresponding

diagram then looks like

As in (46) and (48), the arrows indicate the adjoint actions of the $sl(2)$ generators. The kinked arrows refer to parts of the J_0 actions.

4.3 Modified KZ equations

The logarithmic WZW model hosting the affine Jordan cells introduced above, is based on an extension of the affine Sugawara construction (35). As the actions of the Virasoro modes depend on the target field being a unified cell or not (compare (4) to the first commutator in (2)), the actions of the affine modes appearing in the extended affine Sugawara construction must have a similar dependence. The generalization of (36) thus reads

$$\Delta + \mu = \frac{\kappa^{ab} D_a(x; \theta) D_b(x; \theta)}{2(k+2)} = \frac{(j+\theta)(j+\theta+1)}{k+2} \quad (55)$$

from which it follows that the nilpotent part of the conformal weight, μ , is related to the nilpotent part of the spin, θ , as

$$\mu = \frac{2j+1}{k+2} \theta \quad (56)$$

Likewise, the KZ equations may be extended to cover correlators of generating-function unified cells simply by replacing $D_a(x_i)$ by $D_a(x_i; \theta_i)$ if the i th field is such a unified cell. With the understanding that a non-cellular generating-function primary field, $\Upsilon_i(z_i, x_i; 0)$, corresponds to setting $\theta_i = 0$ in $\Upsilon_i(z_i, x_i; \theta_i)$, the modified KZ equations read

$$0 = K Z_i \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle, \quad i = 1, \dots, \mathcal{N} \quad (57)$$

where

$$KZ_i = \left((k+2)\partial_{z_i} - \sum_{j \neq i} \frac{\kappa^{ab} D_a(x_i; \theta_i) D_b(x_j; \theta_j)}{z_i - z_j} \right) \quad (58)$$

As in the non-logarithmic case, these \mathcal{N} differential equations are not all independent as they satisfy (39). This is true for all combinations of \mathcal{N} generating-function fields, i.e., every generating-function field can be a generating-function primary field or a generating-function Jordan cell.

5 Correlators in logarithmic $SL(2, \mathbb{R})$ WZW models

We now turn to the computation of correlators in the logarithmic WZW model introduced above. Focus will be on two- and three-point chiral blocks of generating-functions. The correlators are worked out as $SL(2, \mathbb{R})$ group-invariant solutions to the conformal Ward identities and are subsequently demonstrated to satisfy the generalized KZ equations (57), (58). This means that conformal and $SL(2, \mathbb{R})$ group invariance fix the form of the two- and three-point chiral blocks as is the case in the ordinary, non-logarithmic $SL(2, \mathbb{R})$ WZW model.

5.1 $SL(2, \mathbb{R})$ group invariance and conformal Ward identities

Bearing the link (56) in mind, the conformal Ward identities (5) now read

$$\begin{aligned} 0 &= \sum_{i=1}^{\mathcal{N}} \partial_{z_i} \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \\ 0 &= \sum_{i=1}^{\mathcal{N}} \left(z_i \partial_{z_i} + \Delta_i + \frac{2j_i + 1}{k+2} \theta_i \right) \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \\ 0 &= \left(\mathcal{L}^{\mathcal{N}} + 2 \sum_{i=1}^{\mathcal{N}} \frac{2j_i + 1}{k+2} \theta_i z_i \right) \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \end{aligned} \quad (59)$$

where the differential operator $\mathcal{L}^{\mathcal{N}}$ is defined in (6). Correlators satisfying these identities are said to be projectively invariant. Likewise, invariance under $SL(2, \mathbb{R})$ group transformations (the ones generated by the horizontal $sl(2)$ algebra) is sometimes referred to as loop-projective invariance. The corresponding Ward identities are often called affine Ward identities, though we will refer to them as $SL(2, \mathbb{R})$ Ward identities. For generating-function correlators involving generating-function Jordan cells (and possibly non-cellular generating-function primary fields) as the ones appearing in (59), they read

$$\begin{aligned} 0 &= \sum_{i=1}^{\mathcal{N}} \partial_{x_i} \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \\ 0 &= \sum_{i=1}^{\mathcal{N}} (x_i \partial_{x_i} - j_i - \theta_i) \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \\ 0 &= \left(\mathcal{J}^{\mathcal{N}} - 2 \sum_{i=1}^{\mathcal{N}} \theta_i x_i \right) \langle \Upsilon_1(z_1, x_1; \theta_1) \dots \Upsilon_{\mathcal{N}}(z_{\mathcal{N}}, x_{\mathcal{N}}; \theta_{\mathcal{N}}) \rangle \end{aligned} \quad (60)$$

Here we have introduced the differential operator

$$\mathcal{J}^{\mathcal{N}} = \sum_{i=1}^{\mathcal{N}} (x_i^2 \partial_{x_i} - 2j_i x_i) \quad (61)$$

It is noted that the middle identities in (59) and (60) follow from the first and third identities. This is a simple consequence of $[L_1, L_{-1}] = 2L_0$ and $[J_+, J_-] = 2J_0$, respectively. The first conformal and $SL(2, \mathbb{R})$ Ward identities merely impose translation invariance on the correlators, allowing us to express them solely in terms of differences, $z_i - z_j$ and $x_i - x_j$, between coordinates of the same type.

The two sets of identities are very similar in nature, as the act of replacing (x_i, j_i, θ_i) by $(z_i, -\Delta_i, -\mu_i)$ in the operators appearing in (60) leads to (59). Also, one of the sets of operators does not depend on the group coordinates x_i , while the other set of operators does not depend on the conformal coordinates z_i . We know that the form of two- and three-point conformal blocks is fixed by the conformal Ward identities, cf. Section 2. We also know that the two- and three-point chiral blocks involving only generating-function primary fields factorize into a ‘conformal’ part and a ‘group’ part. It is therefore natural to expect that the general two- and three-point chiral blocks may factor into a conformal part and a group part. Our analysis will support this assertion. It is demonstrated in the process, though, that one would miss a wealth of solutions by making such a factorization ansatz too naively. Furthermore, a factorization is not guaranteed a priori, so we will base our analysis on very general ansätze for the form of the two- and three-point chiral blocks.

5.2 Two-point chiral blocks

Let us first comment on the factorization ansatz alluded to above. The naive approach would be to consider two-point chiral blocks constructed by multiplying expressions of the form (11) with similar expressions for the group part. To illustrate the shortage of this procedure, we focus on the two-point chiral block of two generating-function unified cells:

$$\begin{aligned} \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \rangle &= \delta_{\Delta_1, \Delta_2} \frac{A^1 (\mu_1 + \mu_2 - 2\mu_1 \mu_2 \ln z_{12}) + A^{12} \mu_1 \mu_2}{z_{12}^{\Delta_1 + \Delta_2}} \\ &\times \delta_{j_1, j_2} \{ B^1 (\theta_1 + \theta_2 + 2\theta_1 \theta_2 \ln x_{12}) - B^{12} \theta_1 \theta_2 \} x_{12}^{j_1 + j_2} \\ &= \delta_{j_1, j_2} \frac{2A^1 B^1 (j_1 + j_2 + 1)}{k + 2} \theta_1 \theta_2 \frac{x_{12}^{j_1 + j_2}}{z_{12}^{\Delta_1 + \Delta_2}} \end{aligned} \quad (62)$$

This could have been the end of the story in which case the only non-vanishing two-point chiral block of the individual fields would have been the one containing two generating-function logarithmic fields. Furthermore, no logarithmic singularity would occur. As we will discuss presently, there are more non-trivial solutions than this one.

We will base our analysis on the following ansatz

$$\langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \rangle = \{ A(\theta_1, \theta_2) + B(\theta_1, \theta_2) \ln z_{12} + C(\theta_1, \theta_2) \ln x_{12} \} \frac{x_{12}^{2s}}{z_{12}^{2h}} \quad (63)$$

where

$$A(\theta_1, \theta_2) = A^0 + A^1 \theta_1 + A^2 \theta_2 + A^{12} \theta_1 \theta_2 \quad (64)$$

and similarly for $B(\theta_1, \theta_2)$ and $C(\theta_1, \theta_2)$. Correlators involving non-cellular primary fields correspond to setting the associated θ s equal to zero. Due to the translational invariance (in both sets of coordinates) of the ansatz, it suffices to impose the two identities involving $\mathcal{L}^{\mathcal{N}}$ and $\mathcal{J}^{\mathcal{N}}$. The third conformal Ward identity (59) thus leads to the conditions

$$\begin{aligned} 0 &= \left(-h + \Delta_1 + \frac{2j_1 + 1}{k + 2} \theta_1 \right) A(\theta_1, \theta_2) + \frac{1}{2} B(\theta_1, \theta_2) \\ 0 &= \left(-h + \Delta_2 + \frac{2j_2 + 1}{k + 2} \theta_2 \right) A(\theta_1, \theta_2) + \frac{1}{2} B(\theta_1, \theta_2) \\ 0 &= \left(-h + \Delta_1 + \frac{2j_1 + 1}{k + 2} \theta_1 \right) B(\theta_1, \theta_2) = \left(-h + \Delta_2 + \frac{2j_2 + 1}{k + 2} \theta_2 \right) B(\theta_1, \theta_2) \end{aligned}$$

$$0 = \left(-h + \Delta_1 + \frac{2j_1 + 1}{k + 2}\theta_1\right) C(\theta_1, \theta_2) = \left(-h + \Delta_2 + \frac{2j_2 + 1}{k + 2}\theta_2\right) C(\theta_1, \theta_2) \quad (65)$$

whereas the third $SL(2, \mathbb{R})$ Ward identity corresponds to the conditions

$$\begin{aligned} 0 &= (s - j_1 - \theta_1) A(\theta_1, \theta_2) + \frac{1}{2} C(\theta_1, \theta_2) = (s - j_2 - \theta_2) A(\theta_1, \theta_2) + \frac{1}{2} C(\theta_1, \theta_2) \\ 0 &= (s - j_1 - \theta_1) B(\theta_1, \theta_2) = (s - j_2 - \theta_2) B(\theta_1, \theta_2) \\ 0 &= (s - j_1 - \theta_1) C(\theta_1, \theta_2) = (s - j_2 - \theta_2) C(\theta_1, \theta_2) \end{aligned} \quad (66)$$

We defer the analysis of these conditions to Appendix A. It is noted, though, that as in the case of (non-affine) conformal Jordan cells [13], one may lose solutions for correlators involving non-cellular fields if one simply sets the corresponding θ s equal to zero in the solution for unified cells only. Instead, examining the conditions case by case (distinguished by the number of generating-function unified cells appearing in the generating-function two-point chiral block) as done in Appendix A results in

$$\begin{aligned} \langle \Upsilon_1(z_1, x_1; 0) \Upsilon_2(z_2, x_2; 0) \rangle &= A^0 W_2 \\ \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; 0) \rangle &= A^1 \theta_1 W_2 \\ \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \rangle \\ &= \left\{ A^1 \theta_1 + A^1 \theta_2 + A^{12} \theta_1 \theta_2 - 2A^1 \theta_1 \theta_2 \left(\frac{j_1 + j_2 + 1}{k + 2} \ln z_{12} - \ln x_{12} \right) \right\} W_2 \end{aligned} \quad (67)$$

Here we have introduced the abbreviation

$$W_2 = \delta_{j_1, j_2} \frac{x_{12}^{j_1 + j_2}}{z_{12}^{\Delta_1 + \Delta_2}} \quad (68)$$

and used that the identity $A^2 = A^1$ is required in the last two-point chiral block. It is recalled that the weights are related to the spins according to (25). In terms of the individual correlators, it follows from (67) that

$$\begin{aligned} \langle \Phi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \rangle &= 0 \\ \langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \rangle &= A^1 W_2 \end{aligned} \quad (69)$$

and

$$\begin{aligned} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle &= 0 \\ \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle &= A^1 W_2 \\ \langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \rangle &= \left\{ A^{12} - 2A^1 \left(\frac{j_1 + j_2 + 1}{k + 2} \ln z_{12} - \ln x_{12} \right) \right\} W_2 \end{aligned} \quad (70)$$

The remaining two-point chiral blocks are obtained by appropriately permuting the indices. It is stressed that the structure constant A^1 appearing in (69) a priori is independent of the structure constant A^1 appearing in (70).

For a translational-invariant two-point chiral block, there is only one independent KZ equation. Referring to the ansatz (63) or to the (loop-)projectively invariant expressions (67), it may be written

$$\begin{aligned} 0 &= \left((k + 2) \partial_{z_{12}} + \frac{-x_{12}^2 \partial_{x_{12}}^2 + 2(j_1 + j_2 + \theta_1 + \theta_2) x_{12} \partial_{x_{12}} - 2(j_1 + \theta_1)(j_2 + \theta_2)}{z_{12}} \right) \\ &\times \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \rangle \end{aligned} \quad (71)$$

Here one or both of the nilpotent parameters may vanish depending on the number of generating-function unified cells there are in the correlator. It is straightforward to verify that the generating-function two-point chiral blocks (67) satisfy this KZ equation.

5.3 Three-point chiral blocks

Here we base our analysis on the ansatz

$$\begin{aligned}
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; \theta_3) \rangle \\
&= \{ A(\theta_1, \theta_2, \theta_3) + B_{12}(\theta_1, \theta_2, \theta_3) \ln z_{12} + B_{23}(\theta_1, \theta_2, \theta_3) \ln z_{23} + B_{13}(\theta_1, \theta_2, \theta_3) \ln z_{13} \\
&\quad + C_{12}(\theta_1, \theta_2, \theta_3) \ln x_{12} + C_{23}(\theta_1, \theta_2, \theta_3) \ln x_{23} + C_{13}(\theta_1, \theta_2, \theta_3) \ln x_{13} \\
&\quad + D_{11}(\theta_1, \theta_2, \theta_3) \ln^2 z_{12} + D_{12}(\theta_1, \theta_2, \theta_3) \ln z_{12} \ln z_{23} + D_{13}(\theta_1, \theta_2, \theta_3) \ln z_{12} \ln z_{13} \\
&\quad + D_{22}(\theta_1, \theta_2, \theta_3) \ln^2 z_{23} + D_{23}(\theta_1, \theta_2, \theta_3) \ln z_{23} \ln z_{13} + D_{33}(\theta_1, \theta_2, \theta_3) \ln^2 z_{13} \\
&\quad + E_{11}(\theta_1, \theta_2, \theta_3) \ln z_{12} \ln x_{12} + E_{12}(\theta_1, \theta_2, \theta_3) \ln z_{12} \ln x_{23} + E_{13}(\theta_1, \theta_2, \theta_3) \ln z_{12} \ln x_{13} \\
&\quad + E_{21}(\theta_1, \theta_2, \theta_3) \ln z_{23} \ln x_{12} + E_{22}(\theta_1, \theta_2, \theta_3) \ln z_{23} \ln x_{23} + E_{23}(\theta_1, \theta_2, \theta_3) \ln z_{23} \ln x_{13} \\
&\quad + E_{31}(\theta_1, \theta_2, \theta_3) \ln z_{13} \ln x_{12} + E_{32}(\theta_1, \theta_2, \theta_3) \ln z_{13} \ln x_{23} + E_{33}(\theta_1, \theta_2, \theta_3) \ln z_{13} \ln x_{13} \\
&\quad + F_{11}(\theta_1, \theta_2, \theta_3) \ln^2 x_{12} + F_{12}(\theta_1, \theta_2, \theta_3) \ln x_{12} \ln x_{23} + F_{13}(\theta_1, \theta_2, \theta_3) \ln x_{12} \ln x_{13} \\
&\quad + F_{22}(\theta_1, \theta_2, \theta_3) \ln^2 x_{23} + F_{23}(\theta_1, \theta_2, \theta_3) \ln x_{23} \ln x_{13} + F_{33}(\theta_1, \theta_2, \theta_3) \ln^2 x_{13} \} \frac{x_{12}^{s_1} x_{23}^{s_2} x_{13}^{s_3}}{z_{12}^{h_1} z_{23}^{h_2} z_{13}^{h_3}} \quad (72)
\end{aligned}$$

where

$$A(\theta_1, \theta_2, \theta_3) = A^0 + A^1 \theta_1 + A^2 \theta_2 + A^3 \theta_3 + A^{12} \theta_1 \theta_2 + A^{23} \theta_2 \theta_3 + A^{13} \theta_1 \theta_3 + A^{123} \theta_1 \theta_2 \theta_3 \quad (73)$$

and similarly for the other θ -dependent structure constants: $B_{ij}(\theta_1, \theta_2, \theta_3)$, $C_{ij}(\theta_1, \theta_2, \theta_3)$, $D_{ij}(\theta_1, \theta_2, \theta_3)$, $E_{ij}(\theta_1, \theta_2, \theta_3)$, and $F_{ij}(\theta_1, \theta_2, \theta_3)$. The conditions following from imposing the conformal and $SL(2, \mathbb{R})$ Ward identities are discussed in Appendix A. This analysis leads to the following generating-function three-point chiral blocks:

$$\begin{aligned}
& \langle \Upsilon_1(z_1, x_1; 0) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle = A^0 W_3 \\
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle \\
&= \left\{ A^0 + A^1 \theta_1 - A^0 \theta_1 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) \right\} W_3 \\
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; 0) \rangle \\
&= \left\{ A^0 + A^1 \theta_1 - A^0 \theta_1 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) \right. \\
&\quad + A^2 \theta_2 - A^0 \theta_2 \left(\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} - \ln \frac{x_{12} x_{23}}{x_{13}} \right) \\
&\quad + A^{12} \theta_1 \theta_2 - A^1 \theta_1 \theta_2 \left(\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} - \ln \frac{x_{12} x_{23}}{x_{13}} \right) - A^2 \theta_1 \theta_2 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) \\
&\quad \left. + A^0 \theta_1 \theta_2 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) \left(\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} - \ln \frac{x_{12} x_{23}}{x_{13}} \right) \right\} W_3 \\
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; \theta_3) \rangle \\
&= \left\{ A^1 \theta_1 + A^2 \theta_2 + A^3 \theta_3 + A^{12} \theta_1 \theta_2 - A^1 \theta_1 \theta_2 \left(\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} - \ln \frac{x_{12} x_{23}}{x_{13}} \right) \right. \\
&\quad - A^2 \theta_1 \theta_2 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) \\
&\quad + A^{23} \theta_2 \theta_3 - A^2 \theta_2 \theta_3 \left(\frac{2j_3 + 1}{k + 2} \ln \frac{z_{23} z_{13}}{z_{12}} - \ln \frac{x_{23} x_{13}}{x_{12}} \right) - A^3 \theta_2 \theta_3 \left(\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} - \ln \frac{x_{12} x_{23}}{x_{13}} \right) \\
&\quad \left. + A^{13} \theta_1 \theta_3 - A^3 \theta_1 \theta_3 \left(\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} - \ln \frac{x_{12} x_{13}}{x_{23}} \right) - A^1 \theta_1 \theta_3 \left(\frac{2j_3 + 1}{k + 2} \ln \frac{z_{23} z_{13}}{z_{12}} - \ln \frac{x_{23} x_{13}}{x_{12}} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + A^{123}\theta_1\theta_2\theta_3 - A^{12}\theta_1\theta_2\theta_3 \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) \\
& - A^{23}\theta_1\theta_2\theta_3 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) - A^{13}\theta_1\theta_2\theta_3 \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \\
& + A^1\theta_1\theta_2\theta_3 \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) \\
& + A^2\theta_1\theta_2\theta_3 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) \\
& + A^3\theta_1\theta_2\theta_3 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \Big\} W_3 \quad (74)
\end{aligned}$$

Here we have introduced the abbreviation

$$W_3 = \frac{x_{12}^{j_1+j_2-j_3} x_{23}^{-j_1+j_2+j_3} x_{13}^{j_1-j_2+j_3}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3} z_{23}^{-\Delta_1+\Delta_2+\Delta_3} z_{13}^{\Delta_1-\Delta_2+\Delta_3}} \quad (75)$$

In terms of individual correlators (besides the one for non-cellular primary fields only, which has been already listed in (74)), we thus have

$$\begin{aligned}
\langle \Phi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle &= A^0 W_3 \\
\langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle &= \left\{ A^1 - A^0 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \right\} W_3 \quad (76)
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle &= A^0 W_3 \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle &= \left\{ A^1 - A^0 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \right\} W_3 \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle &= \left\{ A^{12} - A^1 \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) - A^2 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \right. \\
&\quad \left. + A^0 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \right\} W_3 \quad (77)
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= 0 \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= A^1 W_3 \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= \left\{ A^{12} - A^1 \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) - A^2 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \right\} W_3 \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle &= \left\{ A^{123} - A^{12} \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) - A^{23} \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \right. \\
&\quad - A^{13} \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \\
&\quad \left. + A^1 \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) \right\} W_3
\end{aligned}$$

$$\begin{aligned}
& + A^2 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \left(\frac{2j_3+1}{k+2} \ln \frac{z_{23}z_{13}}{z_{12}} - \ln \frac{x_{23}x_{13}}{x_{12}} \right) \\
& + A^3 \left(\frac{2j_1+1}{k+2} \ln \frac{z_{12}z_{13}}{z_{23}} - \ln \frac{x_{12}x_{13}}{x_{23}} \right) \left(\frac{2j_2+1}{k+2} \ln \frac{z_{12}z_{23}}{z_{13}} - \ln \frac{x_{12}x_{23}}{x_{13}} \right) \Big\} W_3
\end{aligned} \tag{78}$$

The remaining three-point chiral blocks are obtained by appropriately permuting the indices.

For a translational-invariant three-point chiral block, there are two independent KZ equations, cf. (39). Referring to the ansatz (72) or to the (loop-)projectively invariant expressions (74), they may be written

$$0 = KZ_i \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; \theta_3) \rangle, \quad i = 1, 2 \tag{79}$$

where

$$\begin{aligned}
KZ_1 &= (k+2)\partial_{z_1} - \frac{2D_0(x_1, \theta_1)D_0(x_2, \theta_2) + D_+(x_1, \theta_1)D_-(x_2, \theta_2) + D_-(x_1, \theta_1)D_+(x_2, \theta_2)}{z_{12}} \\
&\quad - \frac{2D_0(x_1, \theta_1)D_0(x_3, \theta_3) + D_+(x_1, \theta_1)D_-(x_3, \theta_3) + D_-(x_1, \theta_1)D_+(x_3, \theta_3)}{z_{13}} \\
KZ_2 &= (k+2)\partial_{z_2} + \frac{2D_0(x_1, \theta_1)D_0(x_2, \theta_2) + D_+(x_1, \theta_1)D_-(x_2, \theta_2) + D_-(x_1, \theta_1)D_+(x_2, \theta_2)}{z_{12}} \\
&\quad - \frac{2D_0(x_2, \theta_2)D_0(x_3, \theta_3) + D_+(x_2, \theta_2)D_-(x_3, \theta_3) + D_-(x_2, \theta_2)D_+(x_3, \theta_3)}{z_{23}}
\end{aligned} \tag{80}$$

In these expressions, one, two or all three of the nilpotent parameters may vanish depending on the number of generating-function unified cells there are in the correlator. It is straightforward, though rather tedious, to verify that the generating-function three-point chiral blocks (74) satisfy these KZ equations.

5.4 In terms of spins with nilpotent parts

Here we wish to extend to the logarithmic WZW model the idea put forward in [17] that correlators in logarithmic CFT may be represented compactly by considering conformal weights with nilpotent parts $\hat{\Delta} + \hat{\theta}$. The most general results of this kind for two- and three-point conformal blocks were found in [13] and are given above as (13) and (18). As already indicated, we will here associate the generalized spin $j_i + \theta_i$ to the generating-function unified cell $\Upsilon_i(z_i, x_i; \theta_i)$. The corresponding generalized conformal weight thus reads $\Delta_i + \mu_i$ where $\mu_i = (2j_i + 1)\theta_i/(k+2)$. This allows us to express the generating-function correlators for two- and three-point chiral blocks in the following simple way:

$$\begin{aligned}
\langle \Upsilon_1(z_1, x_1; 0) \Upsilon_2(z_2, x_2; 0) \rangle &= \delta_{j_1, j_2} A^0 \frac{x_{12}^{j_1+j_2}}{z_{12}^{\Delta_1+\Delta_2}} \\
\langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; 0) \rangle &= \delta_{j_1, j_2} A^1 \theta_1 \frac{x_{12}^{(j_1+\theta_1)+j_2}}{z_{12}^{(\Delta_1+\mu_1)+\Delta_2}} \\
\langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \rangle &= \delta_{j_1, j_2} \{ A^1 \theta_1 + A^1 \theta_2 + A^{12} \theta_1 \theta_2 \} \frac{x_{12}^{(j_1+\theta_1)+(j_2+\theta_2)}}{z_{12}^{(\Delta_1+\mu_1)+(\Delta_2+\mu_2)}}
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\langle \Upsilon_1(z_1, x_1; 0) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle &= A^0 \frac{x_{12}^{j_1+j_2-j_3} x_{23}^{-j_1+j_2+j_3} x_{13}^{j_1-j_2+j_3}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3} z_{23}^{-\Delta_1+\Delta_2+\Delta_3} z_{13}^{\Delta_1-\Delta_2+\Delta_3}} \\
\langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle &= \{ A^0 + A^1 \theta_1 \} \\
&\quad \times \frac{x_{12}^{(j_1+\theta_1)+j_2-j_3} x_{23}^{-(j_1+\theta_1)+j_2+j_3} x_{13}^{(j_1+\theta_1)-j_2+j_3}}{z_{12}^{(\Delta_1+\mu_1)+\Delta_2-\Delta_3} z_{23}^{-(\Delta_1+\mu_1)+\Delta_2+\Delta_3} z_{13}^{(\Delta_1+\mu_1)-\Delta_2+\Delta_3}}
\end{aligned}$$

$$\begin{aligned}
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; 0) \rangle = \{A^0 + A^1 \theta_1 + A^2 \theta_2 + A^{12} \theta_1 \theta_2\} \\
& \times \frac{x_{12}^{(j_1+\theta_1)+(j_2+\theta_2)-j_3} x_{23}^{-(j_1+\theta_1)+(j_2+\theta_2)+j_3} x_{13}^{(j_1+\theta_1)-(j_2+\theta_2)+j_3}}{z_{12}^{(\Delta_1+\mu_1)+(\Delta_2+\mu_2)-\Delta_3} z_{23}^{-(\Delta_1+\mu_1)+(\Delta_2+\mu_2)+\Delta_3} z_{13}^{(\Delta_1+\mu_1)-(\Delta_2+\mu_2)+\Delta_3}} \\
& \langle \Upsilon_1(z_1, x_1; \theta_1) \Upsilon_2(z_2, x_2; \theta_2) \Upsilon_3(z_3, x_3; \theta_3) \rangle \\
& = \{A^1 \theta_1 + A^2 \theta_2 + A^3 \theta_3 + A^{12} \theta_1 \theta_2 + A^{23} \theta_2 \theta_3 + A^{13} \theta_1 \theta_3 + A^{123} \theta_1 \theta_2 \theta_3\} \\
& \times \frac{x_{12}^{(j_1+\theta_1)+(j_2+\theta_2)-(j_3+\theta_3)} x_{23}^{-(j_1+\theta_1)+(j_2+\theta_2)+(j_3+\theta_3)} x_{13}^{(j_1+\theta_1)-(j_2+\theta_2)+(j_3+\theta_3)}}{z_{12}^{(\Delta_1+\mu_1)+(\Delta_2+\mu_2)-(\Delta_3+\mu_3)} z_{23}^{-(\Delta_1+\mu_1)+(\Delta_2+\mu_2)+(\Delta_3+\mu_3)} z_{13}^{(\Delta_1+\mu_1)-(\Delta_2+\mu_2)+(\Delta_3+\mu_3)}} \quad (82)
\end{aligned}$$

The remaining combination in (81) and the remaining four combinations in (82) are obtained by appropriate permutations in the indices. Thus confirming our assertion, it follows that the two- and three-point chiral blocks factor into a conformal part and an $SL(2, \mathbb{R})$ group part. The extra degrees of freedom in (81) and (82) compared to the incomplete result (62) are contained in the fact that the θ -dependent structure constants in (81) and (82) do not necessarily factor as in (62). It is noted that the present factorization into a conformal part and an $SL(2, \mathbb{R})$ group part is not evident a priori, while our analysis has demonstrated its validity. These compact representations constitute a significant simplification of the results given above (and derived in Appendix A). The verification of the KZ equations is particularly simple when the correlators are expressed in this way.

5.5 Hierarchical structures for chiral blocks

Based on ideas discussed in [12, 11], it was found in [13] that the conformal blocks involving logarithmic fields may be represented in terms of derivatives with respect to the conformal weights. This was reviewed in Section 2.5. Here we wish to extend this idea to the two- and three-point chiral blocks of the logarithmic WZW model introduced above. It is found that hierarchical structures similar to the ones discussed in Section 2.5 apply in the affine case.

First, it is observed that acting on either W_2 or W_3 , we may substitute derivatives with respect to the spins by multiplicative factors according to

$$\partial_{j_1} = \partial_{j_2} \rightarrow -2 \frac{j_1 + j_2 + 1}{k + 2} \ln z_{12} + 2 \ln x_{12} \quad (83)$$

or

$$\begin{aligned}
\partial_{j_1} & \rightarrow -\frac{2j_1 + 1}{k + 2} \ln \frac{z_{12} z_{13}}{z_{23}} + \ln \frac{x_{12} x_{13}}{x_{23}} \\
\partial_{j_2} & \rightarrow -\frac{2j_2 + 1}{k + 2} \ln \frac{z_{12} z_{23}}{z_{13}} + \ln \frac{x_{12} x_{23}}{x_{13}} \\
\partial_{j_3} & \rightarrow -\frac{2j_3 + 1}{k + 2} \ln \frac{z_{23} z_{13}}{z_{12}} + \ln \frac{x_{23} x_{13}}{x_{12}} \quad (84)
\end{aligned}$$

respectively. This simple observation allows us to represent the correlators involving logarithmic fields as follows:

$$\begin{aligned}
\langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \rangle & = A^1 W_2 \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle & = A^1 W_2 \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \rangle & = (A^{12} + A^2 \partial_{j_1} + A^1 \partial_{j_2}) W_2 \\
\langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle & = (A^1 + A^0 \partial_{j_1}) W_3 \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle & = (A^1 + A^0 \partial_{j_1}) W_3 \\
\langle \Psi_1(z_1, x_2) \Psi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle & = (A^{12} + A^1 \partial_{j_2} + A^2 \partial_{j_1} + A^0 \partial_{j_1} \partial_{j_2}) W_3 \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle & = A^1 W_3
\end{aligned}$$

$$\begin{aligned}
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= (A^{12} + A^2 \partial_{j_1} + A^1 \partial_{j_2}) W_3 \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle &= (A^{123} + A^{23} \partial_{j_1} + A^{13} \partial_{j_2} + A^{12} \partial_{j_3} \\
&\quad + A^3 \partial_{j_1} \partial_{j_2} + A^1 \partial_{j_2} \partial_{j_3} + A^2 \partial_{j_1} \partial_{j_3}) W_3
\end{aligned} \tag{85}$$

in addition to expressions obtained by appropriately permuting the indices. One may therefore represent the correlators hierarchically as

$$\begin{aligned}
\langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \rangle &= A^1 W_2 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \rangle \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle &= A^1 W_2 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \rangle &= A^{12} W_2 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Psi_2(z_2, x_2) \rangle + \partial_{j_2} \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle \\
&\quad - \partial_{j_1} \partial_{j_2} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \rangle
\end{aligned} \tag{86}$$

in the case of two-point chiral blocks, and

$$\begin{aligned}
\langle \Psi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle &= A^1 W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Upsilon_2(z_2, x_2; 0) \Upsilon_3(z_3, x_3; 0) \rangle \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle &= A^1 W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle &= A^{12} W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Psi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle \\
&\quad + \partial_{j_2} \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle \\
&\quad - \partial_{j_1} \partial_{j_2} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Upsilon_3(z_3, x_3; 0) \rangle \\
\langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= A^1 W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle &= A^{12} W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
&\quad + \partial_{j_2} \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
&\quad - \partial_{j_1} \partial_{j_2} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
\langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle &= A^{123} W_3 + \partial_{j_1} \langle \Phi_1(z_1, x_1) \Psi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle \\
&\quad + \partial_{j_2} \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle \\
&\quad + \partial_{j_3} \langle \Psi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
&\quad - \partial_{j_1} \partial_{j_2} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Psi_3(z_3, x_3) \rangle \\
&\quad - \partial_{j_2} \partial_{j_3} \langle \Psi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
&\quad - \partial_{j_1} \partial_{j_3} \langle \Phi_1(z_1, x_1) \Psi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle \\
&\quad + \partial_{j_1} \partial_{j_2} \partial_{j_3} \langle \Phi_1(z_1, x_1) \Phi_2(z_2, x_2) \Phi_3(z_3, x_3) \rangle
\end{aligned} \tag{87}$$

in the case of three-point chiral blocks. As above, the remaining correlators may be obtained by appropriately permuting the indices.

6 Hamiltonian reduction

It is well known that $SL(2, \mathbb{R})$ WZW models may be linked to conformal minimal models via hamiltonian reduction [20, 21, 22, 23]. A precise description of this reduction was given at the level of correlators in [14, 15], while a simple and direct proof of this description was presented in [24, 25] based on [26, 25]. The basic idea in this context is to start with an \mathcal{N} -point chiral block of generating-function primary fields in the affine model, in which case the corresponding \mathcal{N} -point conformal block in the CFT is obtained by setting $x_i = z_i$ for $i = 1, \dots, \mathcal{N}$ [14, 15]. This was refined a bit in [24, 25] where it was discussed how the procedure may be performed in two steps by first setting $x_i = x z_i$ followed by fixing the common proportionality constant to $x = 1$, for example.

Our current situation is quite simple, though, since we are only interested in two- and three-point functions and in their form rather than the relations between structure constants and their dependencies

on the spins and conformal weights. The objective here is therefore to study whether a naive extension of the hamiltonian-reduction principle setting $x_i = z_i$ applies to the logarithmic correlators found above. That is, we wish to show that the two- and three-point chiral blocks (67) and (74) reduce to the two- and three-point conformal blocks (11) and (16), respectively, upon setting $x_i = z_i$. The conformal weights, $\hat{\Delta}_i$, in the resulting logarithmic CFT should then be given by

$$\hat{\Delta}_i = \Delta_i - j_i = \frac{j_i(j_i + 1)}{k + 2} - j_i \quad (88)$$

whereas the central charges are related as $\hat{c} = c - 6k - 2 = \frac{3k}{k+2} - 6k - 2$. It is emphasized that we are only concerned with the form of the correlators, not the various dependencies of the structure constants.

The reductions are straightforward to analyze when the correlators are expressed compactly in terms of spins and conformal weights with nilpotent parts. We thus wish to examine the link between (81) and (82) on one hand and (13) and (18) on the other hand. It follows that the affine correlators reduce to the conformal ones with conformal weights given in (88), if the identifications $x_i = z_i$ for $i = 1, \dots, \mathcal{N}$ are accompanied by

$$\hat{\theta}_i = \mu_i - \theta_i = \left(\frac{2j_i + 1}{k + 2} - 1 \right) \theta_i, \quad i = 1, \dots, \mathcal{N} \quad (89)$$

and (for $j_i, j_{i'} \neq (k + 1)/2$) the renormalizations

$$\begin{aligned} \hat{A}^0 &= A^0 \\ \hat{A}^i &= \frac{A^i}{\frac{2j_i+1}{k+2} - 1}, \quad 1 \leq i \leq 3 \\ \hat{A}^{ii'} &= \frac{A^{ii'}}{\left(\frac{2j_i+1}{k+2} - 1 \right) \left(\frac{2j_{i'}+1}{k+2} - 1 \right)}, \quad 1 \leq i < i' \leq 3 \\ \hat{A}^{123} &= \frac{A^{123}}{\left(\frac{2j_1+1}{k+2} - 1 \right) \left(\frac{2j_2+1}{k+2} - 1 \right) \left(\frac{2j_3+1}{k+2} - 1 \right)} \end{aligned} \quad (90)$$

The apparent subtlety in the case of two-point functions, composed of generating-function unified cells only, is resolved by the Kronecker delta function in j_1 and j_2 appearing in (81).

In the exceptional case where $j_i = (k + 1)/2$, this hamiltonian-reduction procedure corresponds to formally replacing $\Upsilon_i(z_i, x_i; \theta_i)$ by the non-cellular primary field $\Upsilon(z_i; 0)$, cf. (89), in which case the renormalizations (90) involving j_i no longer apply. The representation-theoretical mechanism underlying this reduction in logarithmic nature remains to be understood.

Following [27], primary fields are called *proper primary* if their operator-product expansions with each other cannot produce a logarithmic field. It is argued in [27] (see also [28]) that the structure constants of three-point conformal blocks not involving improper primary fields are related. According to [13] and in the notation used above, such conformal blocks are obtained by setting

$$\hat{A}^1 = \hat{A}^2 = \hat{A}^3, \quad \hat{A}^{12} = \hat{A}^{23} = \hat{A}^{13} \quad (91)$$

This class of restricted three-point conformal blocks can be reached by hamiltonian reduction of a particular subset of the three-point chiral blocks in the affine case. This is quite obvious in the framework with generalized spins and generalized conformal weights, cf. (90). One merely sets

$$\begin{aligned} \frac{A^1}{\frac{2j_1+1}{k+2} - 1} &= \frac{A^2}{\frac{2j_2+1}{k+2} - 1} = \frac{A^3}{\frac{2j_3+1}{k+2} - 1} \\ \frac{A^{12}}{\left(\frac{2j_1+1}{k+2} - 1 \right) \left(\frac{2j_2+1}{k+2} - 1 \right)} &= \frac{A^{23}}{\left(\frac{2j_2+1}{k+2} - 1 \right) \left(\frac{2j_3+1}{k+2} - 1 \right)} = \frac{A^{13}}{\left(\frac{2j_1+1}{k+2} - 1 \right) \left(\frac{2j_3+1}{k+2} - 1 \right)} \end{aligned} \quad (92)$$

in the last chiral block in (82). Hamiltonian reduction then reproduces the last three-point conformal block in (18) with (91) satisfied.

7 Conclusion

We have studied a particular type of logarithmic extension of $SL(2, \mathbb{R})$ WZW models. It is based on the introduction of affine Jordan cells constructed as multiplets of quasi-primary fields organized in indecomposable representations of the Lie algebra $sl(2)$. We have found the general solution to the simultaneously imposed set of conformal and $SL(2, \mathbb{R})$ Ward identities for two- and three-point chiral blocks. These correlators may involve logarithmic terms and may be represented compactly by considering spins with nilpotent parts. The chiral blocks have been found to exhibit hierarchical structures obtained by computing derivatives with respect to the spins. A set of KZ equations, appropriately modified to cover affine Jordan cells, have been derived, and the chiral blocks have been shown to satisfy these equations. It has been also demonstrated that a simple and well-established prescription for hamiltonian reduction at the level of correlators extends straightforwardly to the logarithmic correlators as the latter reduce to the known results for two- and three-point conformal blocks in logarithmic CFT.

We find it natural to say that our results pertain to affine Jordan cells of rank two. This is supported in part by the fact that hamiltonian reduction of the chiral blocks results in correlators of rank-two conformal Jordan cells. In order to argue more directly, we recall that a conformal Jordan cell of rank r [11] consists of one primary field, $\varphi_0(z)$, and $r - 1$ logarithmic and quasi-primary partner fields, $\varphi_1(z), \dots, \varphi_{r-1}(z)$, satisfying

$$[L_n, \varphi_i(z)] = \left(z^{n+1} \partial_z + \hat{\Delta}(n+1)z^n \right) \varphi_i(z) + (n+1)z^n \varphi_{i-1}(z) \quad (93)$$

One could say that the field $\varphi_i(z)$ has degree i or is at depth i . That is, the depth is given by the number of adjoint actions of the Virasoro modes required to reach the primary field in the cell. The rank is then given by one plus the maximum depth. If we extend this characterization to the affine case, we would say that a field is at depth i if i adjoint actions of the Lie algebra generators (or more generally, i adjoint actions of symmetry generators) are required to reach a primary field in the affine Jordan cell. With the rank denoting one plus the maximum depth just defined, the rank of our affine Jordan cells is indeed two.

It would be interesting to extend our work to higher ranks in the sense just indicated. A natural construction seems to suggest itself and is based on the following simple observation. The conformal Jordan cell (93) may be written compactly as

$$[L_n, v(z; \hat{\theta})] = \left(z^{n+1} \partial_z + (\hat{\Delta} + \hat{\theta})(n+1)z^n \right) v(z) \quad (94)$$

where we have introduced the generating-function unified cell as

$$v(z; \hat{\theta}) = \sum_{i=0}^{r-1} \hat{\theta}^i \varphi_i(z) \quad (95)$$

In this Section 7, $\hat{\theta}$ is a nilpotent, yet even, parameter satisfying

$$\hat{\theta}^r = 0, \quad \hat{\theta}^{r-1} \neq 0 \quad (96)$$

We thus suggest to generalize the affine Jordan cell by introducing the differential-operator realization

$$\begin{aligned} D_+(x; \theta) &= x^2 \partial_x - 2(j + \theta)x \\ D_0(x; \theta) &= x \partial_x - (j + \theta) \\ D_-(x; \theta) &= -\partial_x \end{aligned} \quad (97)$$

of the Lie algebra $sl(2)$ and the corresponding generating-function unified cell $v(z, x; \theta)$ satisfying

$$[J_a, v(z, x; \theta)] = -D_a(x; \theta)v(z, x; \theta) \quad (98)$$

In this Section 7, θ is a nilpotent, yet even, parameter satisfying

$$\theta^r = 0, \quad \theta^{r-1} \neq 0 \quad (99)$$

The generalization of the expansion (42) would then read

$$v(z, x; \theta) = \sum_{i=0}^{r-1} \theta^i \Theta_i(z, x) \quad (100)$$

where $\Theta_0(z, x)$ is a generating-function primary field similar to $\Phi(z, x)$ in (42). An examination of hamiltonian reduction of the correlators based on these higher-rank affine Jordan cells require knowledge on higher-rank conformal Jordan cells. Partial results in this direction may be found in [11, 12]. Conformal Jordan cells of infinite rank have been introduced in [29].

An interesting extension of the work [13] concerns the general solution to the superconformal Ward identities appearing in logarithmic superconformal field theory. Results in this direction may be found in [30]. A complete solution would facilitate an extension of the present work to $OSp(1|2)$ WZW models and their hamiltonian reduction. This deserves to be explored further.

As already mentioned, we hope to address elsewhere the classification problem of affine Jordan cells. In particular, indecomposable representations as extensions of non-integrable representations would be interesting to understand. These results could eventually be extended further to the higher-rank affine Jordan cells based on (97) and (98) and could be developed along the lines of Section 4.

We also hope to study the four-point chiral blocks involving our affine Jordan cells. It appears straightforward to implement the Ward identities, after which the general four-point chiral blocks should follow from the modified KZ equations (57), (58). To test whether the extended prescription for hamiltonian reduction employed above also applies to these four-point functions, one could compare the resulting correlators to the recently obtained results on four-point conformal blocks [31, 32].

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A Analysis of Ward identities

Below are indicated some of the steps leading to the general expressions for the generating-function two- and three-point chiral blocks given in (81) and (82), respectively.

A.1 Two-point chiral blocks

We initially consider the case with two generating-function unified cells, that is, $\theta_1, \theta_2 \neq 0$. Expanding (65) leads to the conditions

$$\begin{aligned} 0 &= B^0 - 2(h - \Delta_1)A^0 = B^0 - 2(h - \Delta_2)A^0 \\ 0 &= B^1 - 2(h - \Delta_1)A^1 + 2\frac{2j_1 + 1}{k + 2}A^0 = B^1 - 2(h - \Delta_2)A^1 \\ 0 &= B^2 - 2(h - \Delta_1)A^2 = B^2 - 2(h - \Delta_2)A^2 + 2\frac{2j_2 + 1}{k + 2}A^0 \\ 0 &= B^{12} - 2(h - \Delta_1)A^{12} + 2\frac{2j_1 + 1}{k + 2}A^2 = B^{12} - 2(h - \Delta_2)A^{12} + 2\frac{2j_2 + 1}{k + 2}A^1 \end{aligned}$$

$$\begin{aligned}
0 &= (h - \Delta_1)B^0 = (h - \Delta_2)B^0 \\
0 &= (h - \Delta_1)B^1 - \frac{2j_1 + 1}{k + 2}B^0 = (h - \Delta_2)B^1 \\
0 &= (h - \Delta_1)B^2 = (h - \Delta_2)B^2 - \frac{2j_2 + 1}{k + 2}B^0 \\
0 &= (h - \Delta_1)B^{12} - \frac{2j_1 + 1}{k + 2}B^2 = (h - \Delta_2)B^{12} - \frac{2j_2 + 1}{k + 2}B^1 \\
0 &= (h - \Delta_1)C^0 = (h - \Delta_2)C^0 \\
0 &= (h - \Delta_1)C^1 - \frac{2j_1 + 1}{k + 2}C^0 = (h - \Delta_2)C^1 \\
0 &= (h - \Delta_1)C^2 = (h - \Delta_2)C^2 - \frac{2j_2 + 1}{k + 2}C^0 \\
0 &= (h - \Delta_1)C^{12} - \frac{2j_1 + 1}{k + 2}C^2 = (h - \Delta_2)C^{12} - \frac{2j_2 + 1}{k + 2}C^1
\end{aligned} \tag{101}$$

whereas an expansion of (66) yields the conditions

$$\begin{aligned}
0 &= C^0 + 2(s - j_1)A^0 = C^0 + 2(s - j_2)A^0 \\
0 &= C^1 + 2(s - j_1)A^1 - 2A^0 = C^1 + 2(s - j_2)A^1 \\
0 &= C^2 + 2(s - j_1)A^2 = C^2 + 2(s - j_2)A^2 - 2A^0 \\
0 &= C^{12} + 2(s - j_1)A^{12} - 2A^2 = C^{12} + 2(s - j_2)A^{12} - 2A^1 \\
0 &= (s - j_1)B^0 = (s - j_2)B^0 \\
0 &= (s - j_1)B^1 - B^0 = (s - j_2)B^1 \\
0 &= (s - j_1)B^2 = (s - j_2)B^2 - B^0 \\
0 &= (s - j_1)B^{12} - B^2 = (s - j_2)B^{12} - B^1 \\
0 &= (s - j_1)C^0 = (s - j_2)C^0 \\
0 &= (s - j_1)C^1 - C^0 = (s - j_2)C^1 \\
0 &= (s - j_1)C^2 = (s - j_2)C^2 - C^0 \\
0 &= (s - j_1)C^{12} - C^2 = (s - j_2)C^{12} - C^1
\end{aligned} \tag{102}$$

It follows immediately that a non-trivial solution requires

$$s = j_1 = j_2, \quad h = \Delta_1 = \Delta_2 \tag{103}$$

further implying the relations

$$\begin{aligned}
0 &= A^0 = B^0 = B^1 = B^2 = C^0 = C^1 = C^2 \\
0 &= A^1 - A^2 = B^{12} + 2\frac{2j_1 + 1}{k + 2}A^1 = C^{12} - 2A^1
\end{aligned} \tag{104}$$

The parameter A^{12} is independent of the other ones.

In the case where $\theta_1 \neq 0$ while $\theta_2 = 0$, the third conformal Ward identity (i.e., (65)) yields

$$\begin{aligned}
0 &= B^0 - 2(h - \Delta_1)A^0 = B^0 - 2(h - \Delta_2)A^0 \\
0 &= B^1 - 2(h - \Delta_1)A^1 + 2\frac{2j_1 + 1}{k + 2}A^0 = B^1 - 2(h - \Delta_2)A^1 \\
0 &= (h - \Delta_1)B^0 = (h - \Delta_2)B^0 \\
0 &= (h - \Delta_1)B^1 - \frac{2j_1 + 1}{k + 2}B^0 = (h - \Delta_2)B^1
\end{aligned}$$

$$\begin{aligned}
0 &= (h - \Delta_1)C^0 = (h - \Delta_2)C^0 \\
0 &= (h - \Delta_1)C^1 - \frac{2j_1 + 1}{k + 2}C^0 = (h - \Delta_2)C^1
\end{aligned} \tag{105}$$

while the third $SL(2, \mathbb{R})$ Ward identity (i.e., (66)) corresponds to

$$\begin{aligned}
0 &= C^0 + 2(s - j_1)A^0 = C^0 + 2(s - j_2)A^0 \\
0 &= C^1 + 2(s - j_1)A^1 - 2A^0 = C^1 + 2(s - j_2)A^1 \\
0 &= (s - j_1)B^0 = (s - j_2)B^0 \\
0 &= (s - j_1)B^1 - B^0 = (s - j_2)B^1 \\
0 &= (s - j_1)C^0 = (s - j_2)C^0 \\
0 &= (s - j_1)C^1 - C^0 = (s - j_2)C^1
\end{aligned} \tag{106}$$

It is stressed that B^2 , for example, does not exist (or is set to zero) in this case and should therefore not be treated as a free parameter. As above, the spins and weights are seen to satisfy (103), and it follows that

$$0 = A^0 = B^0 = B^1 = C^0 = C^1 \tag{107}$$

while A^1 is the only free parameter.

In the case where $\theta_1 = \theta_2 = 0$, the two sets of conditions reduce to

$$\begin{aligned}
0 &= B^0 - 2(h - \Delta_1)A^0 = B^0 - 2(h - \Delta_2)A^0 \\
0 &= (h - \Delta_1)B^0 = (h - \Delta_2)B^0 \\
0 &= (h - \Delta_1)C^0 = (h - \Delta_2)C^0
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
0 &= C^0 + 2(s - j_1)A^0 = C^0 + 2(s - j_2)A^0 \\
0 &= (s - j_1)B^0 = (s - j_2)B^0 \\
0 &= (s - j_1)C^0 = (s - j_2)C^0
\end{aligned} \tag{109}$$

Once again, the spins and weights satisfy (103). This time, $B^0 = C^0 = 0$ while A^0 is the only free parameter.

This analysis leads to the two-point chiral blocks given in (67).

A.2 Three-point chiral blocks

Based on the ansatz (72), the third conformal Ward identity (59) corresponds to the conditions

$$\begin{aligned}
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)A + C_{12} + C_{13} = (s_1 + s_2 - 2j_2 - 2\theta_2)A + C_{12} + C_{23} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)A + C_{23} + C_{13} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)B_{12} + E_{11} + E_{13} = (s_1 + s_2 - 2j_2 - 2\theta_2)B_{12} + E_{11} + E_{12} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)B_{12} + E_{12} + E_{13} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)B_{23} + E_{21} + E_{23} = (s_1 + s_2 - 2j_2 - 2\theta_2)B_{23} + E_{21} + E_{22} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)B_{23} + E_{22} + E_{23} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)B_{13} + E_{31} + E_{33} = (s_1 + s_2 - 2j_2 - 2\theta_2)B_{13} + E_{31} + E_{32} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)B_{13} + E_{32} + E_{33} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)C_{12} + 2F_{11} + F_{13} = (s_1 + s_2 - 2j_2 - 2\theta_2)C_{12} + 2F_{11} + F_{12} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)C_{12} + F_{12} + F_{13}
\end{aligned}$$

$$\begin{aligned}
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)C_{23} + F_{12} + F_{23} = (s_1 + s_2 - 2j_2 - 2\theta_2)C_{23} + F_{12} + 2F_{22} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)C_{23} + 2F_{22} + F_{23} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)C_{13} + F_{13} + 2F_{33} = (s_1 + s_2 - 2j_2 - 2\theta_2)C_{13} + F_{13} + F_{23} \\
&= (s_2 + s_3 - 2j_3 - 2\theta_3)C_{13} + F_{23} + 2F_{33} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)D_{ij} = (s_1 + s_2 - 2j_2 - 2\theta_2)D_{ij} = (s_2 + s_3 - 2j_3 - 2\theta_3)D_{ij} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)E_{ij} = (s_1 + s_2 - 2j_2 - 2\theta_2)E_{ij} = (s_2 + s_3 - 2j_3 - 2\theta_3)E_{ij} \\
0 &= (s_1 + s_3 - 2j_1 - 2\theta_1)F_{ij} = (s_1 + s_2 - 2j_2 - 2\theta_2)F_{ij} = (s_2 + s_3 - 2j_3 - 2\theta_3)F_{ij} \quad (110)
\end{aligned}$$

whereas the third $SL(2, \mathbb{R})$ Ward identity corresponds to the conditions

$$\begin{aligned}
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)A + B_{12} + B_{13} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)A + B_{12} + B_{23} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)A + B_{23} + B_{13} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)B_{12} + 2D_{11} + D_{13} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)B_{12} + 2D_{11} + D_{12} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)B_{12} + D_{12} + D_{13} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)B_{23} + D_{12} + D_{23} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)B_{23} + D_{12} + 2D_{22} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)B_{23} + 2D_{22} + D_{23} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)B_{13} + D_{13} + 2D_{33} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)B_{13} + D_{13} + D_{23} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)B_{13} + D_{23} + 2D_{33} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)C_{12} + E_{11} + E_{31} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)C_{12} + E_{11} + E_{21} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)C_{12} + E_{21} + E_{31} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)C_{23} + E_{12} + E_{32} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)C_{23} + E_{12} + E_{22} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)C_{23} + E_{22} + E_{32} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)C_{13} + E_{13} + E_{33} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)C_{13} + E_{13} + E_{23} \\
&= (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)C_{13} + E_{23} + E_{33} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)D_{ij} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)D_{ij} = (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)D_{ij} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)E_{ij} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)E_{ij} = (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)E_{ij} \\
0 &= (-h_1 - h_3 + 2\Delta_1 + 2\mu_1)F_{ij} = (-h_1 - h_2 + 2\Delta_2 + 2\mu_2)F_{ij} = (-h_2 - h_3 + 2\Delta_3 + 2\mu_3)F_{ij} \quad (111)
\end{aligned}$$

To keep the notation simple, we have left out the explicit indications that the structure constants depend on the θ s. To keep the presentation simple as well, we will leave out most of the details in the analysis of these conditions. Since the approach essentially is the same as the one employed in the study of two-point chiral blocks, we will merely outline the main steps.

We distinguish between the different numbers of unified cells, that is, the different numbers of non-vanishing θ s. In every case, one finds the relations

$$\begin{aligned}
s_1 &= j_1 + j_2 - j_3, & s_2 &= -j_1 + j_2 + j_3, & s_3 &= j_1 - j_2 + j_3 \\
h_1 &= \Delta_1 + \Delta_2 - \Delta_3, & h_2 &= -\Delta_1 + \Delta_2 + \Delta_3, & h_3 &= \Delta_1 - \Delta_2 + \Delta_3 \quad (112)
\end{aligned}$$

Having split the analysis into the four cases characterized by 0, 1, 2 or 3 unified cells, one expands the conditions (110) and (111) on the set of associated θ s, where it is recalled that $\mu_i = (2j_i + 1)\theta_i/(k + 2)$. Since the resulting conditions are linear in the structure constants A^0, B_{ij}^0 etc, it is straightforward to work out the relations between the structure constants associated to the four cases. The relations are listed below.

In the case where $\theta_1 = \theta_2 = \theta_3 = 0$, we find

$$B_{ij}^0 = C_{ij}^0 = D_{ij}^0 = E_{ij}^0 = F_{ij}^0 = 0 \quad (113)$$

which means that A^0 is the only free structure constant.

In the case where $\theta_2 = \theta_3 = 0$ while $\theta_1 \neq 0$, we find

$$\begin{aligned} 0 &= B_{ij}^0 = C_{ij}^0 = D_{ij}^0 = D_{ij}^1 = E_{ij}^0 = E_{ij}^1 = F_{ij}^0 = F_{ij}^1 \\ B_{12}^1 &= -B_{23}^1 = B_{13}^1 = -\frac{2j_1+1}{k+2}A^0 \\ C_{12}^1 &= -C_{23}^1 = C_{13}^1 = A^0 \end{aligned} \quad (114)$$

while A^1 is unconstrained. That is, we may consider A^0 and A^1 as the only independent structure constants.

In the case where $\theta_3 = 0$ while $\theta_1, \theta_2 \neq 0$, we find

$$\begin{aligned} 0 &= B_{ij}^0 = C_{ij}^0 = D_{ij}^0 = D_{ij}^1 = D_{ij}^2 = E_{ij}^0 = E_{ij}^1 = E_{ij}^2 = F_{ij}^0 = F_{ij}^1 = F_{ij}^2 \\ B_{12}^1 &= -B_{23}^1 = B_{13}^1 = -\frac{2j_1+1}{k+2}A^0, \quad B_{12}^2 = B_{23}^2 = -B_{13}^2 = -\frac{2j_2+1}{k+2}A^0 \\ B_{12}^{12} &= -\frac{2j_2+1}{k+2}A^1 - \frac{2j_1+1}{k+2}A^2, \quad B_{23}^{12} = -B_{13}^{12} = -\frac{2j_2+1}{k+2}A^1 + \frac{2j_1+1}{k+2}A^2 \\ C_{12}^1 &= -C_{23}^1 = C_{13}^1 = A^0, \quad C_{12}^2 = C_{23}^2 = -C_{13}^2 = A^0 \\ C_{12}^{12} &= A^1 + A^2, \quad C_{23}^{12} = -C_{13}^{12} = A^1 - A^2 \\ D_{11}^{12} &= -D_{22}^{12} = \frac{1}{2}D_{23}^{12} = -D_{33}^{12} = \frac{(2j_1+1)(2j_2+1)}{(k+2)^2}A^0, \quad D_{12}^{12} = D_{13}^{12} = 0 \\ E_{11}^{12} &= -E_{22}^{12} = E_{23}^{12} = E_{32}^{12} = -E_{33}^{12} = -2\frac{j_1+j_2+1}{k+2}A^0 \\ E_{12}^{12} &= -E_{13}^{12} = -E_{21}^{12} = E_{31}^{12} = \frac{-2j_1+2j_2}{k+2}A^0 \\ F_{11}^{12} &= -F_{22}^{12} = \frac{1}{2}F_{23}^{12} = -F_{33}^{12} = A^0, \quad F_{12}^{12} = F_{13}^{12} = 0 \end{aligned} \quad (115)$$

while A^{12} is unconstrained. That is, we may consider A^0 , A^1 , A^2 and A^{12} as the only independent structure constants.

In the case where $\theta_1, \theta_2, \theta_3 \neq 0$, we find

$$\begin{aligned} 0 &= A^0 = B_{ij}^0 = B_{ij}^l = C_{ij}^0 = C_{ij}^l \\ 0 &= D_{ij}^0 = D_{ij}^l = D_{ij}^{lm} = E_{ij}^0 = E_{ij}^l = E_{ij}^{lm} = F_{ij}^0 = F_{ij}^l = F_{ij}^{lm} \\ B_{12}^{12} &= -\frac{2j_2+1}{k+2}A^1 - \frac{2j_1+1}{k+2}A^2, \quad B_{23}^{12} = -B_{13}^{12} = -\frac{2j_2+1}{k+2}A^1 + \frac{2j_1+1}{k+2}A^2 \\ B_{23}^{23} &= -\frac{2j_3+1}{k+2}A^2 - \frac{2j_2+1}{k+2}A^3, \quad B_{12}^{23} = -B_{13}^{23} = \frac{2j_3+1}{k+2}A^2 - \frac{2j_2+1}{k+2}A^3 \\ B_{13}^{13} &= -\frac{2j_3+1}{k+2}A^1 - \frac{2j_1+1}{k+2}A^3, \quad B_{12}^{13} = -B_{23}^{13} = \frac{2j_3+1}{k+2}A^1 - \frac{2j_1+1}{k+2}A^3 \\ B_{12}^{123} &= \frac{2j_3+1}{k+2}A^{12} - \frac{2j_1+1}{k+2}A^{23} - \frac{2j_2+1}{k+2}A^{13} \\ B_{23}^{123} &= -\frac{2j_3+1}{k+2}A^{12} + \frac{2j_1+1}{k+2}A^{23} - \frac{2j_2+1}{k+2}A^{13} \\ B_{12}^{233} &= -\frac{2j_3+1}{k+2}A^{12} - \frac{2j_1+1}{k+2}A^{23} + \frac{2j_2+1}{k+2}A^{13} \\ C_{12}^{12} &= A^1 + A^2, \quad C_{23}^{12} = -C_{13}^{12} = A^1 - A^2 \\ C_{23}^{23} &= A^2 + A^3, \quad C_{12}^{23} = -C_{13}^{23} = -A^2 + A^3 \\ C_{13}^{13} &= A^1 + A^3, \quad C_{12}^{13} = -C_{23}^{13} = -A^1 + A^3 \end{aligned}$$

$$\begin{aligned}
C_{12}^{123} &= -A^{12} + A^{23} + A^{13}, & C_{23}^{123} &= A^{12} - A^{23} + A^{13}, & C_{13}^{123} &= A^{12} + A^{23} - A^{13} \\
D_{11}^{123} &= -\frac{(2j_2+1)(2j_3+1)}{(k+2)^2}A^1 - \frac{(2j_1+1)(2j_3+1)}{(k+2)^2}A^2 + \frac{(2j_1+1)(2j_2+1)}{(k+2)^2}A^3 \\
D_{22}^{123} &= \frac{(2j_2+1)(2j_3+1)}{(k+2)^2}A^1 - \frac{(2j_1+1)(2j_3+1)}{(k+2)^2}A^2 - \frac{(2j_1+1)(2j_2+1)}{(k+2)^2}A^3 \\
D_{33}^{123} &= -\frac{(2j_2+1)(2j_3+1)}{(k+2)^2}A^1 + \frac{(2j_1+1)(2j_3+1)}{(k+2)^2}A^2 - \frac{(2j_1+1)(2j_2+1)}{(k+2)^2}A^3 \\
D_{12}^{123} &= 2\frac{(2j_1+1)(2j_3+1)}{(k+2)^2}A^2, & D_{23}^{123} &= 2\frac{(2j_1+1)(2j_2+1)}{(k+2)^2}A^3 \\
D_{13}^{123} &= 2\frac{(2j_2+1)(2j_3+1)}{(k+2)^2}A^1 \\
E_{11}^{123} &= \frac{2(j_2+j_3+1)}{k+2}A^1 + \frac{2(j_1+j_3+1)}{k+2}A^2 - \frac{2(j_1+j_2+1)}{k+2}A^3 \\
E_{12}^{123} &= \frac{-2j_2+2j_3}{k+2}A^1 - \frac{2(j_1+j_3+1)}{k+2}A^2 + \frac{-2j_1+2j_2}{k+2}A^3 \\
E_{13}^{123} &= -\frac{2(j_2+j_3+1)}{k+2}A^1 + \frac{-2j_1+2j_3}{k+2}A^2 + \frac{2j_1-2j_2}{k+2}A^3 \\
E_{21}^{123} &= \frac{2j_2-2j_3}{k+2}A^1 - \frac{2(j_1+j_3+1)}{k+2}A^2 + \frac{2j_1-2j_2}{k+2}A^3 \\
E_{22}^{123} &= -\frac{2(j_2+j_3+1)}{k+2}A^1 + \frac{2(j_1+j_3+1)}{k+2}A^2 + \frac{2(j_1+j_2+1)}{k+2}A^3 \\
E_{23}^{123} &= \frac{-2j_2+2j_3}{k+2}A^1 + \frac{2j_1-2j_3}{k+2}A^2 - \frac{2(j_1+j_2+1)}{k+2}A^3 \\
E_{31}^{123} &= -\frac{2(j_2+j_3+1)}{k+2}A^1 + \frac{2j_1-2j_3}{k+2}A^2 + \frac{-2j_1+2j_2}{k+2}A^3 \\
E_{32}^{123} &= \frac{2j_2-2j_3}{k+2}A^1 + \frac{-2j_1+2j_3}{k+2}A^2 - \frac{2(j_1+j_2+1)}{k+2}A^3 \\
E_{33}^{123} &= \frac{2(j_2+j_3+1)}{k+2}A^1 - \frac{2(j_1+j_3+1)}{k+2}A^2 + \frac{2(j_1+j_2+1)}{k+2}A^3 \\
F_{11}^{123} &= -A^1 - A^2 + A^3, & F_{22}^{123} &= A^1 - A^2 - A^3, & F_{33}^{123} &= -A^1 + A^2 - A^3 \\
F_{12}^{123} &= 2A^2, & F_{23}^{123} &= 2A^3, & F_{13}^{123} &= 2A^1
\end{aligned} \tag{116}$$

while A^{123} is unconstrained. That is, we may consider A^0 , A^i , A^{ij} and A^{123} as the only independent structure constants.

The relations corresponding to the situation where $\theta_1 = \theta_3 = 0$ while $\theta_2 \neq 0$, for example, are obtained from the relations corresponding to the case where $\theta_2 = \theta_3 = 0$ while $\theta_1 \neq 0$ by an appropriate permutation in the indices.

These results lead to the three-point chiral blocks given in (82).

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